Existence and uniqueness of Ordinary Differential Equation

Once we are given a differential equation, naturally we would like to consider the following basic questions.

1. Is there any solution(s)? (Existence)
2. If yes, is it unique or many solutions? (Uniqueness)

The solution to IVP does not necessarily to be unique. For example, \( y \equiv 0 \) and \( y = x^5 \) are the solutions of

\[
\begin{align*}
\frac{dy}{dx} &= 5y^4 \\
y(0) &= 0
\end{align*}
\]

3. If a differential equation does have a solution, can we find it?

A solution to a differential equation exists does not mean that we will be able to find it. In reality, most of differential equations, we are not able to find the analytical solution (Discouraging, right? Excited since it means there is a lot to learn, right?).

In this course, we focus on question 3: Solving differential equations (Here I mean analytical solution, not numerical solution) if it is possible. As I mentioned above, we can only solve a tiny subset of differential equations, but those equations are very common and useful to be used to model problems. However, we will answer the first two questions for very special and simple case. Here, we will expose to the very basic theorem on existence and uniqueness of first order ODE (with initial value), basically, under a very simple (easily verified) condition (which is a strong condition too, many mathematicians have been looking for weaker condition to guarantee the existence and uniqueness), the first order ODE has a solution and only one solution.

**Existence and uniqueness theorem (EUT):** Suppose that \( f(x, y) \) and \( \frac{\partial f}{\partial y} \) both are continuous in a rectangle

\[
R = \{(x, y) : x_0 - \delta < x < x_0 + \delta, y_0 - \varepsilon < y < y_0 + \varepsilon\}, \text{ where } \delta \text{ and } \varepsilon \text{ are positive.}
\]

Then there exists a number \( \delta_1 > 0 \) such that there is one and only one solution to the following first order ODE:

\[
\begin{align*}
\frac{dy}{dx} &= f(x, y) \\
y(x_0) &= y_0
\end{align*}
\]

for \( x_0 - \delta_1 < x < x_0 + \delta_1 \).
Example 1: Using EUT, to determine whether the following 1st order ODE has a solution or not. If yes, is it unique?

\[
\begin{align*}
    y' &= x - y + 3 \\
    y(0) &= 1
\end{align*}
\]

In this case, both the function \( f(x, y) = x - y + 3 \) and \( \frac{\partial f}{\partial y}(x, y) = -1 \) are continuous at all points \((x, y)\). Thus, by EUT, there is one and only one solution.

Example 2: Using EUT, to determine whether \( \frac{dy}{dx} = 5y^3 \) has unique solution?

\[
\begin{align*}
    y(0) &= 0
\end{align*}
\]

It is clear that \( f(x, y) = 5y^3 \) is continuous at any points, but \( \frac{\partial f}{\partial y}(x, y) = \frac{4}{\sqrt[3]{y}} \) is not continuous when \( y = 0 \). Thus \( \frac{\partial f}{\partial y}(x, y) \) is not continuous for any rectangle contains \((0,0)\). In this case, EUT does not apply. In other words, just by EUT, we do not know whether there is a unique solution.

Remark: There are two solutions: \( y \equiv 0 \) and \( y = x^5 \) to this initial value problem.

Example 3: Using EUT, to determine whether \( \frac{dy}{dx} = 5y^3 \) has unique solution?

\[
\begin{align*}
    y(0) &= 1
\end{align*}
\]

Clearly, \( f(x, y) = 5y^3 \) and \( \frac{\partial f}{\partial y}(x, y) = \frac{4}{3\sqrt[3]{y}} \) are continuous in a rectangle \( R = \{(x, y) : -\frac{1}{2} < x < \frac{1}{2}, \frac{1}{2} < y < \frac{3}{2}\} \) which contains \((0,1)\), so there is one and only one solution to the above IVP by EUT.

Theorem 2: If the functions \( p \) and \( g \) are continuous on an open interval \( I = (\alpha, \beta) \) containing the point \( t = t_0 \), then there exists a unique function \( y = \phi(t) \) that satisfies the differential equation

\[ y' + p(t)y = g(t) \]

for each \( t \in [\alpha, \beta] \), and that also satisfies the initial condition \( y(t_0) = y_0 \), where \( y_0 \) is an arbitrary prescribed initial value.

Example 4: Use Theorem 2 to find an interval in which the initial value problem
\[
\begin{align*}
\begin{cases}
y' + 2y = 4t^2 \\
y(0) = 3
\end{cases}
\end{align*}
\]
has a unique solution.

We rewrite \( ny' + 2y = 4t^2 \) to \( y' + \frac{2}{n} y = 4t \), so \( p(t) = \frac{2}{n} \), \( g(t) = 4t \). Thus, \( p(t) \) is continuous on \( (-\infty, 0) \cup (0, +\infty) \) and \( g(t) \) is continuous everywhere. The interval \( (0, +\infty) \) contains the initial point, consequently, Theorem 2 guarantees that the above initial value problem has a unique solution on the interval \( (0, +\infty) \).