5.5 Orthonormal Sets

In working with an inner product spaces V, it is generally desirable to have a basis of mutually orthogonal unit vectors. This is convenient not only in finding coordinates of vectors, but also in solving least squares problems.

Definition 1: Let $v_1, v_2, \cdots, v_n$ be vectors in an inner product space V. If $<v_i, v_j>=0$ when $i \neq j$, then $\{v_1, v_2, \cdots, v_n\}$ is called **orthogonal set** of vectors.

Property 1: $\{v_1, v_2, \cdots, v_n\}$ is a nonzero vector orthogonal set , then $v_1, v_2, \cdots, v_n$ are linearly independent.

Definition 2: An orthonormal set of vectors in an orthogonal set of unit vectors.

Example 1: For the vector space $C[-\pi, \pi]$, we define an inner product

$$< f, g >= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$$

Verify $<\frac{1}{\sqrt{2}}, \cos x, \cos 2x, \cdots, \cos nx>$ is an orthonormal set.

Solution: $<\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}>=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}dx=1$

$$<\cos x, \cos x>=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos x\cos x dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\cos 2x)dx=1$$

$$<\cos nx, \cos nx>=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos nx\cos nx dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\cos 2nx)dx=1$$

$$<\cos ix, \cos jx>=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos ix\cos jx dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}(\cos(i+j)x+\cos(i-j)x)dx=0 \ (i \neq j)$$

Exercise 1: Verify $v_1=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, v_2=\frac{1}{\sqrt{14}}\begin{pmatrix}2\\1\\-3\end{pmatrix}, v_3=\frac{1}{\sqrt{42}}\begin{pmatrix}4\\-5\\1\end{pmatrix}$ forms an orthonormal set.

Property 2: $\{v_1, v_2, \cdots, v_n\}$ is an orthonormal basis for an inner product vector space V.

If $v=c_1v_1+c_2v_2+\cdots c_nv_n$, then $c_i=<v, v_i>$.

Proof:

$$<v_i, v>=<v_i, c_1v_1+c_2v_2++c_iv_i+\cdots c_nv_n>=c_1<v_i, v_1>+c_2<v_i, v_2>+\cdots c_i<v_i, v_i>+c_n<v_i, v_n>=c_i$$

Property 3: $\{v_1, v_2, \cdots, v_n\}$ is an orthonormal basis for an inner product vector space V, and $u = \sum_{i=1}^{n} a_i v_i$, $v = \sum_{i=1}^{n} b_i v_i$, then $< u, v >= \sum_{i=1}^{n} a_i b_i$

Property 4 (Parseval's Formula): $\{v_1, v_2, \cdots, v_n\}$ is an orthonormal basis for an inner product vector space V, and $v = \sum_{i=1}^{n} c_i v_i$, then $\| v \|^2 = \sum_{i=1}^{n} c_i^{\,2}$

Example 2: Given that $\{\frac{1}{\sqrt{2}}, \cos 2x\}$ is an orthonormal set in $C[-\pi, \pi]$ with inner product as in Example 1, determine the value of $\int_{-\pi}^{\pi} \sin^4 x dx$ without computing antiderivatives.

Solution: Since $\sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} - \frac{1}{2}\cos 2x$, it follows from Parseval's formula that

$$\int_{-\pi}^{\pi} \sin^4 x dx = \pi \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^4 x dx = \pi \| \sin^2 x \|^2 = \pi [(\frac{1}{\sqrt{2}})^2 + (-\frac{1}{2})^2] = \frac{3}{4}\pi$$

Definition: An $n \times n$ matrix Q is said to be an **orthogonal matrix** if the column vectors of Q from an orthonormal set in R^n.

Properties of orthogonal matrix:

(i). The column vectors of Q form an orthonormal basis for R^n.

(ii). $Q^T Q = I$

(iii). $Q^T = Q^{-1}$

(iv). $< Qx, Qy >=< x, y >$

(v). $\| Qx \|_2 = \| x \|_2$

HW: 1,4, 6