1.1. Systems of Linear Equations

Definition: **A linear equation** in n unknowns is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \cdots, a_n, b$ are constants, while $x_1, x_2, \cdots, x_n$ are variables.

Definition: **A linear system of m equations in n unknowns (**also called $m \times n$ system) is a system of the form

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \qquad (1)$$

where a_{ij}, b_i, are constants, while $x_1, x_2, \cdots, x_n$ are variables.

Definition: **A solution of the system** is an n-vector satisfying all of the equation.

A $m \times n$ system can have (1) no solution, (2) only one solution, (3) infinite solutions.

Example: Identify the solution(s) for the following systems:

(a) $\begin{cases} x + y = 1 \\ x + y = 0 \end{cases}$

(b) $\begin{cases} 6x + 6y = 6 \\ 2x + 2y = 2 \end{cases}$

(c) $\begin{cases} x + y = 1 \\ x - y = 0 \end{cases}$

Interpretation geometrically?

Definition: **Equivalent systems** means two systems have same solution set.

Example: Justify the systems are equivalent or not?

(a) system 1: $\begin{array}{c} x + y = 1 \\ x - y = 0 \end{array}$

system 2: $\begin{array}{c} x - y = 0 \\ x + y = 1 \end{array}$

(b). system 1: $\begin{array}{c} x + y = 1 \\ x - y = 0 \end{array}$

system 2: $\begin{array}{c} 5x + 5y = 5 \\ x - y = 0 \end{array}$

(c). system 1: $\begin{array}{c} x + y = 1 \\ x - y = 0 \end{array}$

system 2: $\begin{array}{c} x + y = 1 \\ 3x + y = 2 \end{array}$

There are three operations that can be used to obtain an equivalent system:

I. Interchanging two equations.
II. Multiplying both sides by a nonzero number.
III. Adding a multiple of one equation to another equation.

Definition: **Triangular system** means in the kth equation the coefficients of the first $k-1$ variables are all zero and the coefficient of x_k is nonzero ($k = 1,2,\cdots,n$).

Example: The following system is a strict triangular system:

$$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 4 \\ 2x_2 - x_3 &= 3 \\ -8x_3 &= 8 \end{aligned}$$

(a) Solve the above system.
(b) How to solve it efficiently? (Backwards substitution.)

Now, you can see that if the system is a triangular system, you can quickly obtain the solution. Thus, we can try to use the three operations (I,II and III) to transform the original system into a triangular system.

Using matrix multiplication, we can write $m \times n$ system (1) into the following matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad (A|B) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

A is called the coefficient matrix, $(A\,|\,B)$ is called the augmented matrix.

The system can be solved by performing operations on the augmented matrix. Corresponding to the three operations used to obtain equivalent systems, the following row operations may be applied to the augmented matrix.

Elementary Row Operations:

I. Interchange two rows ($R_i \leftrightarrow R_j$)

II. Multiply a row by a nonzero number ($aR_i \to R_i (a \neq 0)$)

III. Replace a row by its sum with a multiple of another row ($aR_i + R_j \to R_j$)

We can use the above three elementary row operations to transform a matrix to a triangular matrix and obtain the solution.

Example: Using elementary row operation to solve the following equation.

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 4 \\ -2x_1 + x_2 = -1 \\ x_1 - 2x_2 + x_3 = -2 \end{cases}$$

Get the augmented matrix first.

$$(A\,|\,B) = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 4 \\ -2 & 1 & 0 & -1 \\ 1 & -2 & 1 & -2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -2 \\ -2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 4 \end{array}\right]$$

$$\xrightarrow[-3R_1 + R_3 \to R_3]{2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -2 \\ 0 & -3 & 2 & -5 \\ 0 & 8 & -2 & 10 \end{array}\right] \xrightarrow{\frac{8}{3}R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -2 \\ 0 & -3 & 2 & -5 \\ 0 & 0 & \frac{10}{3} & -\frac{10}{3} \end{array}\right]$$

Then we use backwards substitution to get the solution is $x_1 = 1, x_2 = 1, x_3 = -1$.
Or you can continue to use row operation to obtain the solution.

$$\xrightarrow{\frac{3}{10}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -2 \\ 0 & -3 & 2 & -5 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[-R_3 + R_1 \to R_1]{-2R_3 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{-\frac{1}{3}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{2R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

This is clear that the solution is $x_1 = 1, x_2 = 1, x_3 = -1$.

Homework: 1.1 1. (c) 5. (c). 6. (f) 9.