1.2 Row Echelon Form

Definition: A Linear system is **overdetermined** if it has more equations than variables.

Example: The following system is an overdetermined system:

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 - x_2 = 1 \\ x_1 - 2x_2 = 0 \end{cases}$$

No solution to the above system.

Definition: A Linear system is **underdetermined** if it has less equations than variables.

Example: The following system is an underdetermined system:

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 - x_3 = 1 \end{cases}$$

Infinite solutions to the above system.

Definition: A Linear system is **square** if the number of equations as variables.

Example: The following system is a square system:

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 - x_2 = 1 \end{cases}$$

One solution to the above system.

Definition: A system of equations is **consistent** if it has a solution.

In general, an overdetermined system is generally not consistent, an underdetermined system is general consistent, a square system is generally consistent has one solution.

Definition: A system of linear equations is said to be homogeneous if the constants on the right-hand side are all zero.

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \\ x_1 - 2x_3 = 0 \end{cases}$$

The above is a homogeneous system. Is there any solution to the above system? ($x_1 = x_2 = x_3 = 0$ is a solution)

Properties of Homogeneous System:

(a). Homogeneous systems are always consistent (either trivial solution ($x_1 = x_2 = \cdots = x_n = 0$) or infinite solution).

(b). An $m \times n$ homogeneous system of linear equations has a nontrivial solution if $m < n$.

Definition: **Leading zeros of a row** of a matrix means a sequence of zeros of a row which starts at the first entry:

Example: (0,0,0,0,5,0,2,0) has four leading zeros.

Definition: A variable corresponding to the first nonzero entry of a nonzero row is called a **lead variable**. The other variable is called **free variable**.

Definition: A matrix is said to be in **row echelon form** if

(i) The first nonzero entry in each row is 1.
(ii) If row k does not consist entirely of zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zero entries in row k.
(iii) If there are rows whose entries are all zero, they are below the rows having nonzero entries.

Example: The following matrices are in row echelon form:

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Example: The following matrices are not in row echelon form:

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The first matrix does not satisfy condition (i); the second does not satisfy (ii); the third one does not satisfy (iii).

Definition: A matrix is said to be in **reduced row echelon form** if:

(i) The matrix is in row echelon form
(ii) The first nonzero entry in each row is the only nonzero entry in its column.

Example: Determine which of the following matrices are not in reduced row echelon form.

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The first, third and fourth matrices are not in reduced echelon form since they do not satisfy condition (ii).

Last section, we have learned that to solve linear systems by transferring the system to triangular system. This section, we introduce the Gaussian elimination by transferring the system's augmented matrix in row echelon form and Gauss-Jordan reduction method by transferring the system's augmented matrix in reduced row echelon form to solve linear system. Procedures are similar.

Example: Solve the following equation:

$$\begin{cases} -x_1 + x_2 - x_3 + 3x_4 = 0 \\ 3x_1 + x_2 - x_3 - x_4 = 0 \\ 2x_1 - x_2 - 2x_3 - x_4 = 0 \end{cases}$$

$$(A|B) = \left[\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right] \xrightarrow{-R_1 \to R_1} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[-2R_1+R_3 \to R_3]{-3R_1+R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right] \xrightarrow{\frac{1}{4}R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{array}\right] \xrightarrow{-\frac{1}{3}R_3 \to R_3} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[-R_3+R_1 \to R_1]{R_3+R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2+R_1 \to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

Let x_4 be any number $x_4 = t$. We have the solution: $\begin{cases} x_1 = t \\ x_2 = -t \\ x_3 = t \\ x_4 = t \end{cases}$

Example: Consider a linear system whose augmented matrix is of the form

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & \alpha & 3 \end{array}\right]$$

For what values of α will the system have a unique solution?

$$(A|B)=\left[\begin{array}{ccc|c}1 & 2 & 1 & 1\\ -1 & 4 & 3 & 2\\ 2 & -2 & \alpha & 3\end{array}\right]\xrightarrow[-2R_1+R_3\to R_3]{R_1+R_2\to R_2}\left[\begin{array}{ccc|c}1 & 2 & 1 & 1\\ 0 & 6 & 4 & 3\\ 0 & -6 & \alpha-2 & 1\end{array}\right]\xrightarrow{R_2+R_3\to R_3}\left[\begin{array}{ccc|c}1 & 2 & 1 & 1\\ 0 & 6 & 4 & 3\\ 0 & 0 & \alpha+2 & 4\end{array}\right]$$

As long as $\alpha+2\neq 0, \alpha\neq -2$, the system has a unique solution.

Example: Solve the following equation:

$$\begin{cases} x_1+3x_2+x_3+x_4=3\\ 2x_1-2x_2+x_3+2x_4=8\\ 3x_1+x_2+2x_3+3x_4=11\end{cases}$$

$$(A|B)=\left[\begin{array}{cccc|c}1 & 3 & 1 & 1 & 3\\ 2 & -2 & 1 & 2 & 8\\ 3 & 1 & 2 & 3 & 11\end{array}\right]\xrightarrow[-3R_1+R_3\to R_3]{-2R_1+R_2\to R_2}\left[\begin{array}{cccc|c}1 & 3 & 1 & 1 & 3\\ 0 & -8 & -1 & 0 & 2\\ 0 & -8 & -1 & 0 & 2\end{array}\right]\xrightarrow{-\frac{1}{8}R_2\to R_2}\left[\begin{array}{cccc|c}1 & 3 & 1 & 1 & 3\\ 0 & 1 & \frac{1}{8} & 0 & -\frac{1}{4}\\ 0 & -8 & -1 & 0 & 2\end{array}\right]$$

$$\xrightarrow{8R_2+R_3\to R_3}\left[\begin{array}{cccc|c}1 & 3 & 1 & 1 & 3\\ 0 & 1 & \frac{1}{8} & 0 & -\frac{1}{4}\\ 0 & 0 & 0 & 0 & 0\end{array}\right]\xrightarrow{-3R_2+R_1\to R_1}\left[\begin{array}{cccc|c}1 & 0 & \frac{5}{8} & 1 & \frac{15}{4}\\ 0 & 1 & \frac{1}{8} & 0 & -\frac{1}{4}\\ 0 & 0 & 0 & 0 & 0\end{array}\right]$$

Let x_3, x_4 be any numbers $x_3=t, x_4=s$. We have the solution: $\begin{cases} x_1=\frac{15}{4}-\frac{5}{8}t-s\\ x_2=-\frac{1}{4}-\frac{1}{8}t\\ x_3=t\\ x_4=s\end{cases}$

Homework: 1, 2(a), (c),(f), 3, 5(e)-(j), 6(c),(d), 9.