1.3 Matrix Algebra

A **matrix** is a rectangular array of numbers arranged in rows and columns.

An m x n (this is often called the **size** or **dimension** of the matrix) matrix is a matrix with m rows and n columns and the element in the i^{th} row and j^{th} column is denoted by a_{ij} . A general m x n matrix is the following:

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} = (a_{ij})_{m\times n}\,.$$

The size or dimension of a matrix is subscripted as shown if required. If it's not required or clear from the problem the subscripted size is often dropped from the matrix.

Special matrices:

Square matrix has same number of rows and columns.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

Diagonal matrix is a square matrix with only zero entries off the main diagonal (that is $a_{ij} = 0$ if $i \neq j$):

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Identity matrix: $I = I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$

Zero matrix: $$0_{m\times n} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{m\times n}$$

Column matrix: $$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}$$

Row matrix: $y = (y_1, y_2, \cdots, y_n)_{1\times n}$

we often refer to row matrix and column matrix as vectors.

Matrix addition, subtraction and Scalar multiplication:

$$A \pm B = (a_{ij})_{m\times n} \pm (b_{ij})_{m\times n} = (a_{ij} \pm b_{ij})_{m\times n}$$
$$cA = c(a_{ij})_{m\times n} = (ca_{ij})_{m\times n} \ (c \text{ is a number})$$

Example 1: Given the following two matrices:

$$A = \begin{pmatrix} 2 & 5 \\ -1 & 6 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$$

Compute *2A-3B.*

Matrix Multiplication:

Definition 1: Product of a $1\times n$ row matrix and an $n\times 1$ column matrix is a 1×1 matrix given by

$$(y_1, y_2, \cdots, y_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)$$

Example 2: Find the product $\begin{pmatrix} 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 2\times1 + 0\times2 + (-1)\times3 = -1$$

Definition 2: If *A* is an $m \times p$ matrix and *B* is a $p \times n$ matrix, then the **matrix product** of *A* and *B*, denoted by *AB*, is an $m \times n$ matrix whose element in the *i*th row and *j*th column is the number obtained from the product of the *i*th row of *A* and *j*th column of *B*. If the number of columns in *A* does not equal the number of rows in *B*, then the matrix product *AB* is not defined.

Example 3: Find the matrix product $\begin{pmatrix} 2 & 3 & -1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -2 & 0 \\ -1 & 2 \end{pmatrix}$

$$\begin{pmatrix} 2 & 3 & -1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -2 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2\times1 + 3\times(-2) + (-1)\times(-1) & 2\times(-3) + 3\times0 + (-1)\times2 \\ -2\times1 + 1\times(-2) + 2\times(-1) & -2\times(-3) + 1\times0 + 2\times2 \end{pmatrix} = \begin{pmatrix} -3 & -8 \\ -6 & 10 \end{pmatrix}$$

Example 4: Given $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, compute *AB* and *BA*. Are they same?

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Matrix Inverse

Definition 3: For a given square matrix *A* if we can find another matrix of the same size *B*, such that $AB = BA = I$, then we call *A* is **invertible** or **nonsingular**, and B the inverse of *A* and denote it by $B = A^{-1}$, so $AA^{-1} = A^{-1}A = I$. A matrix that has no inverse is said to be **singular**.

Example 5: Given $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Prove that $A^{-1} = A$.

Since $AA=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $A^{-1}=A$

Definition 4: The **transpose** of $m\times n$ matrix A is the $n\times m$ matrix B defined by $a_{ij}=b_{ji}$. The transpose of A is denoted by A^T

$$A=(a_{ij})_{m\times n}=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&&&\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix}_{m\times n}\quad B=(b_{ij})_{n\times m}=\begin{bmatrix}a_{11}&a_{21}&\cdots&a_{m1}\\a_{12}&a_{22}&\cdots&a_{m2}\\\vdots&&&\\a_{1n}&a_{2n}&\cdots&a_{mn}\end{bmatrix}_{n\times m}=A^T$$

Example 6: $A=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$, then $A^T=\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$

Property of transpose:

i. $(A^T)^T=A$
ii. $(\alpha A)^T=\alpha A^T$
iii. $(A+B)^T=A^T+B^T$
iv. $(AB)^T=B^TA^T$

Definition: An $n\times n$ matrix A is said to be **symmetric** if $A=A^T$

Example 7: $A=\begin{pmatrix}1&2&3\\2&5&6\\3&8&9\end{pmatrix}$ is not symmetric, $A=\begin{pmatrix}1&2&3\\2&5&8\\3&8&9\end{pmatrix}$ is symmetric.

Definition: If $a_1=\begin{pmatrix}a_{11}\\a_{21}\\\vdots\\a_{m1}\end{pmatrix}, a_2=\begin{pmatrix}a_{12}\\a_{22}\\\vdots\\a_{mn}\end{pmatrix},\cdots,a_n=\begin{pmatrix}a_{1n}\\a_{2n}\\\vdots\\a_{mn}\end{pmatrix}$ are vectors and $c_1,c_2,\cdots,c_n$ are numbers, then the following

$$a_1c_1+a_2c_2+\cdots+a_nc_n$$

is called a **linear combination** of the vectors $a_1,a_2,\cdots,a_n$

Example: Linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

can write as the following matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

We have $Ax = b$.

Also, we can write as the following matrix form:

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} x_2 + \cdots + \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} x_n = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

Thus, we have the following property:

A linear system $Ax = b$ is consistent (means it has at least one solution) if and only if b can be written as a linear combination of the column vectors of A.

Example: If A be a nonsingular matrix. Show that A^T is also nonsingular and $(A^T)^{-1} = (A^{-1})^T$

Prove:
$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$
$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

so, $(A^T)^{-1} = (A^{-1})^T$

Homework: 1. (d),(g),(h), 2, (e),(f), 5, (c), 7, (b), 10, 13, 15

You may use MATLAB to check your matrix computation.

Example 1:

```
>> A=[2 5;-1 6]

A =

     2     5
    -1     6

>> B=[0 2;-1 0]

B =

     0     2
    -1     0

>> 2*A-3*B

ans =

     4     4
     1    12
```

Example 3:

```
>> A=[2 3 -1;-2 1 2]

A =

     2     3    -1
    -2     1     2

>> B=[1 -3;-2 0;-1 2]

B =

     1    -3
    -2     0
    -1     2

>> A*B
```

```
ans =

   -3    -8
   -6    10
```

Example 6:

```
>> A=[0 1 2;1 1 2;2 1 1]

A =

     0     1     2
     1     1     2
     2     1     1

>> inv(A)

ans =

    -1     1     0
     3    -4     2
    -1     2    -1
```

Example 7:

```
>> A=[1 -2;2 -4]

A =

     1    -2
     2    -4

>> inv(A)
Warning: Matrix is singular to working precision.

ans =

   Inf   Inf
   Inf   Inf
```