1.4 Elementary Matrix

There are three row operations on a matrix:

I. Interchanging two rows.
II. Multiplying a row by a nonzero number.
III. Adding a multiple of one row to another row.

The matrix results from exact one row operation acting on identity matrix is called **elementary matrix**.

Example 1: $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & 0 & 1 \end{bmatrix}$

E_1, E_2, E_3 all are elementary matrix.

Example 2: Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, E_1, E_2, E_3 from Example 1, then compute:

$E_1A, E_2A, E_3A, AE_1, AE_2, AE_3$, and figure out the role of the elementary matrix times a matrix from right and left.

$$E_1A = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, E_2A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ -5a_{21} & -5a_{22} & -5a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, E_3A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ -6a_{11} + a_{31} & -6a_{21} + a_{32} & -6a_{31} + a_{33} \end{bmatrix}$$

$$AE_1 = \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix}, AE_2 = \begin{bmatrix} a_{11} & -5a_{12} & a_{13} \\ a_{21} & -5a_{22} & a_{23} \\ a_{31} & -5a_{32} & a_{33} \end{bmatrix}, AE_3 = \begin{bmatrix} a_{11} & a_{12} & a_{11} - 6a_{13} \\ a_{21} & a_{22} & a_{21} - 6a_{23} \\ a_{31} & a_{32} & a_{31} - 6a_{33} \end{bmatrix}$$

Example 3: Let

$$E_1=\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix},\alpha\neq 0,E_2=\begin{pmatrix}1&&&&&&0\\&\ddots&&&&&\\&&1&&&&\\&&&\alpha&&&\\&&&&1&&\\&&&&&\ddots&\\0&&&&&&1\end{pmatrix},E_3=\begin{pmatrix}1&&&&&&0\\&\ddots&&&&&\\&&1&&&&\\&&&\frac{1}{\alpha}&&&\\&&&&1&&\\&&&&&\ddots&\\0&&&&&&1\end{pmatrix},$$

$$E_5=\begin{pmatrix}1&&&&&&0\\\vdots&\ddots&&&&&\\0&\cdots&1&&&&\\\vdots&&&&&&\\0&\cdots&\alpha&\cdots&1&&\\\vdots&&&&&\ddots&\\0&\cdots&0&\cdots&0&\cdots&1\end{pmatrix},E_6=\begin{pmatrix}1&&&&&&0\\\vdots&\ddots&&&&&\\0&\cdots&1&&&&\\\vdots&&&&&&\\0&\cdots&-\alpha&\cdots&1&&\\\vdots&&&&&\ddots&\\0&\cdots&0&\cdots&0&\cdots&1\end{pmatrix}.$$

Please check the following identities by matrix multiplications:

$E_1E_1=I$, $E_3E_2=E_2E_3=I$, $E_6E_5=E_5E_6=I$

So, $E_1^{-1}=E_1, E_2^{-1}=E_3, E_5^{-1}=E_6$

What you can say about the elementary matrix' inverse matrix in general.

Definition: A matrix B it **row equivalent** to A if there exists a finite sequence $E_1,E_2,\cdots,E_k$ of elementary matrices such that

$$B=E_kE_{k-1}\cdots E_1A$$

Exercise 1*: Prove $(E_kE_{k-1}\cdots E_1)^{-1}=E_1^{-1}E_2^{-1}\cdots E_k^{-1}$

$$(E_kE_{k-1}\cdots E_2E_1)(E_1^{-1}E_2^{-1}\cdots E_k^{-1})=E_kE_{k-1}\cdots E_2E_1E_1^{-1}E_2^{-1}\cdots E_{k-1}^{-1}E_k^{-1}=E_kE_{k-1}\cdots E_2IE_2^{-1}\cdots E_{k-1}^{-1}E_k^{-1}$$
$$=E_kE_{k-1}\cdots E_2E_2^{-1}\cdots E_k^{-1}=\cdots=E_kE_{k-1}\cdots I\cdots E_{k-1}^{-1}E_k^{-1}=E_kE_{k-1}E_{k-1}^{-1}E_k^{-1}=E_kE_k^{-1}=I$$
$$(E_1^{-1}E_2^{-1}\cdots E_{k-1}^{-1}E_k^{-1})(E_kE_{k-1}\cdots E_2E_1)=E_1^{-1}E_2^{-1}\cdots E_{k-1}^{-1}E_k^{-1}E_kE_{k-1}\cdots E_2E_1=E_1^{-1}E_2^{-1}\cdots E_{k-1}^{-1}IE_{k-1}\cdots E_2E_1$$
$$=E_1^{-1}E_2^{-1}\cdots E_{k-1}^{-1}E_{k-1}\cdots E_2E_1=E_1^{-1}E_2^{-1}\cdots I\cdots E_2E_1=\cdots=E_1^{-1}E_2^{-1}E_2E_1=E_1^{-1}IE_1=E_1^{-1}E_1=I$$

so, $(E_kE_{k-1}\cdots E_1)^{-1}=E_1^{-1}E_2^{-1}\cdots E_k^{-1}$

Remark: From the above result, we have $(AB)^{-1} = B^{-1}A^{-1}$

Exercise 2: Let $E_k E_{k-1} \cdots E_1 A = I$, prove $A^{-1} = E_k E_{k-1} \cdots E_1$

$E_k E_{k-1} \cdots E_1 A A^{-1} = I A^{-1} \Rightarrow E_k E_{k-1} \cdots E_1 = A^{-1}$

Theorem: Let A be a square matrix. The following are equivalent:

(1). A is nonsingular (or A has an inverse matrix).

(2). $Ax = 0$ has only the trivial solution 0 .

(3). A is row equivalent to I .

Corollary: Let A be a square matrix. If A is nonsingular, then there is a unique solution to system

$$Ax = b$$

The unique solution is $x = A^{-1}b$

Let A be a nonsingular square matrix.

Consider the augmented matrix:

$$(A|I) \xrightarrow{series \quad row \quad operations (E_k E_{k-1} \cdots E_1)} (E_k E_{k-1} \cdots E_1 A | E_k E_{k-1} \cdots E_1) = (I | E_k E_{k-1} \cdots E_1)$$

Through row operations, we transform the left side matrix to identity matrix, the right side is the inverse matrix of A (From the above result of Exercise 2, we have $E_k E_{k-1} \cdots E_1 = A^{-1}$).

Example 4: Let $A = \begin{bmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{bmatrix}$. Find A^{-1}.

$$(A|I) = \left[\begin{array}{ccc|ccc} -1 & -3 & -3 & 1 & 0 & 0 \\ 2 & 6 & 1 & 0 & 1 & 0 \\ 3 & 8 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & -1 & 0 & 0 \\ 2 & 6 & 1 & 0 & 1 & 0 \\ 3 & 8 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-3R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & -1 & 0 & 0 \\ 0 & 0 & -5 & 2 & 1 & 0 \\ 0 & -1 & -6 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & -1 & 0 & 0 \\ 0 & -1 & -6 & 3 & 0 & 1 \\ 0 & 0 & -5 & 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{5}R_3\to R_3]{-R_2\to R_2}\left[\begin{array}{ccc|ccc}1 & 3 & 3 & -1 & 0 & 0\\ 0 & 1 & 6 & -3 & 0 & -1\\ 0 & 0 & 1 & -0.4 & -0.2 & 0\end{array}\right]\xrightarrow[-6R_3+R_2\to R_2]{-3R_3+R_1\to R_1}\left[\begin{array}{ccc|ccc}1 & 3 & 0 & 0.2 & .6 & 0\\ 0 & 1 & 0 & -.6 & 1.2 & -1\\ 0 & 0 & 1 & -0.4 & -0.2 & 0\end{array}\right]$$

$$\xrightarrow{-3R_2+R_1\to R_1}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 2 & -3 & 3\\ 0 & 1 & 0 & -0.6 & 1.2 & -1\\ 0 & 0 & 1 & -0.4 & -0.2 & 0\end{array}\right]$$

$$A^{-1}=\begin{bmatrix}2 & -3 & 3\\ -0.6 & 1.2 & -1\\ -0.4 & -0.2 & 0\end{bmatrix}$$

Exercise 3: Solve the following system

$$\begin{cases}-x_1-3x_2-3x_3=5\\ 2x_1+6x_2+x_3=10\\ 3x_1+8x_2+3x_3=-1\end{cases}$$

Solution: The coefficient matrix is $A=\begin{bmatrix}-1 & -3 & -3\\ 2 & 6 & 1\\ 3 & 8 & 3\end{bmatrix}$, let $b=\begin{pmatrix}5\\ 10\\ -1\end{pmatrix}$, $x=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$

$$Ax=b$$

$$x=A^{-1}b=\begin{bmatrix}2 & -3 & 3\\ -0.6 & 1.2 & -1\\ -0.4 & -0.2 & 0\end{bmatrix}\begin{pmatrix}5\\ 10\\ -1\end{pmatrix}=\begin{pmatrix}-23\\ 10\\ -4\end{pmatrix}$$

Exercise 4: Let $A=\begin{bmatrix}-1 & -3 & -3\\ 2 & 6 & 1\\ 3 & 8 & 3\end{bmatrix}$, $B=\begin{bmatrix}2 & 0 & -3\\ -2 & 6 & 0\\ 1 & 0 & 1\end{bmatrix}$.

(1). Find X such that $AX=B$

(2). Find Y such that $YA=B$

Solution: (1). Find inverse of A. $A^{-1}=\begin{pmatrix} 2 & -3 & 3 \\ -0.6 & 1.2 & -1 \\ -0.4 & -0.2 & 0 \end{pmatrix}$, then compute

$X=A^{-1}B=\begin{pmatrix} 13 & -18 & -3 \\ -4.6 & 7.2 & 0.8 \\ -0.4 & -1.2 & 1.2 \end{pmatrix}$. (2). $Y=BA^{-1}=\begin{pmatrix} 5.2 & -5.4 & 6 \\ -7.6 & 13.2 & 12 \\ 1.6 & -3.2 & 3 \end{pmatrix}$

Exercise 5*: Let $A=\begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix}$.

(1). Find elementary matrixes $E_1, E_2, \cdots, E_k$ such that

$$E_k E_{k-1} \cdots E_1 A = U$$

where U is an upper triangular matrix.

(2). Let $L=E_1^{-1}\cdots E_{k-1}^{-1}E_k^{-1}$. What type of matrix of L (lower or upper triangular matrix)? Thus we have $A=E_1^{-1}\cdots E_{k-1}^{-1}E_k^{-1}U=LU$ (we call this is triangular factorization).

$$(A\,|\,I)=\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 6 & 4 & 5 & 0 & 1 & 0 \\ 4 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-2R_1+R_3\to R_3]{-3R_1+R_2\to R_2} \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3 & 1 & 0 \\ 0 & -1 & 1 & -2 & 0 & 1 \end{array}\right)$$

$E_1=\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_2=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

$$\xrightarrow{R_2+R_3\to R_3} \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 3 & -5 & 1 & 1 \end{array}\right)$$

$E_3=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

We have $U = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

From the Example 3, we have

$$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

So, $L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$

Thus we have $A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ (we call this is triangular factorization).

Homework: 1-5, 7, 12, 16, 19, 28,29