2.1 &2.2 Determinant of Matrix

The determinant is actually a function that takes a **square** matrix and converts it into a number. The standard notation for the determinant of the matrix A is

$$\det(A) = |A|$$

We start to define determinant of 1×1 matrix $A = (a)$. For this 1×1 matrix $A = (a)$, the determinant of matrix $A = (a)$ is defined as

$$\det(A) = |A| = a$$

For 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant of matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined as

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The actual formula for the function is somewhat complex. We need to introduce cofactor to define determinant of any square matrix.

Definition: Let $A = (a_{ij})$ be an $n\times n (n > 1)$ matrix, and let M_{ij} denote the $(n-1)\times(n-1)$ matrix obtained from A by deleting the row and column containing a_{ij}. The determinant of M_{ij} is called the **minor** of a_{ij}. We define the **cofactor** A_{ij} of a_{ij} by

$$A_{ij} = (-1)^{i+j} \det(M_{ij})$$

Example 1: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Find cofactor A_{32}

$$A_{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = -1(a_{11}a_{23} - a_{13}a_{21}) = -(a_{11}a_{23} - a_{13}a_{21})$$

Definition: Let $A = (a_{ij})$ be an $n\times n (n > 1)$ matrix. The determinant of matrix A, denoted by $\det(A) = |A|$ is defined as follows:

$$\det(A) = |A| = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

In the above definition, we used the first row for the cofactor expansion. In fact, we can use any row or column, and we have the following property:

Property: For $A=(a_{ij})$ be an $n\times n(n>1)$ matrix, we have

$$\det(A)=|A|=a_{i1}A_{i1}+a_{i2}A_{i2}+\cdots+a_{in}A_{in}=a_{1j}A_{1j}+a_{2j}A_{2j}+\cdots+a_{nj}A_{nj}$$

for $i=1,2,\cdots n,\ j=1,2,\cdots n.$

Example 2: Let $A=\begin{pmatrix}2&1&1\\0&1&2\\0&3&1\end{pmatrix}$. Compute $\det A=|A|$

$$\det A=\begin{vmatrix}2&1&1\\0&1&2\\0&3&1\end{vmatrix}=2(-1)^{(1+1)}\begin{vmatrix}1&2\\3&1\end{vmatrix}+0(-1)^{(2+1)}\begin{vmatrix}1&1\\3&1\end{vmatrix}+0(-1)^{(3+1)}\begin{vmatrix}1&1\\1&2\end{vmatrix}=2(1-6)=-10$$

Example 3: Compute $A=\begin{pmatrix}9&2&5&0\\5&2&2&1\\4&0&0&0\\1&2&3&2\end{pmatrix}$

$$\det A=\begin{vmatrix}9&2&5&0\\5&2&2&1\\4&0&0&0\\1&2&3&2\end{vmatrix}=4(-1)^{3+1}\begin{vmatrix}2&5&0\\2&2&1\\2&3&2\end{vmatrix}$$

$$=4(2(-1)^{1+1}\begin{vmatrix}2&1\\3&2\end{vmatrix}+5(-1)^{1+2}\begin{vmatrix}2&1\\2&2\end{vmatrix}+0)=4(2(4-3)-5(4-2))=-32$$

From the above example, we can often save work by expanding along the row or column that contains the most zeros.

From the definition, we can derive the following **properties** of matrix determinant:

a. $\det A^T=\det A$
b. If a whole row (or column) of a square matrix A is 0, then $\det A=0$

c. $$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

d. $$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ \alpha a_{i1} & \alpha a_{i2} & \cdots & \alpha a_{in} \\ \vdots & & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \alpha \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

e. $$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ \alpha a_{i1} + a_{j1} & \alpha a_{i2} + a_{j2} & \cdots & \alpha a_{in} + a_{jn} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

f. If a square matrix A has two identical row or column, then $\det A = 0$

g. For a triangular matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & 0 & \ddots & \\ & & & a_{nn} \end{pmatrix}$ or $A = \begin{bmatrix} a_{11} & & & \\ a_{12} & a_{22} & 0 & \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$, then

$\det(A) = a_{11}a_{22}\cdots a_{nn}$

h. $\det(AB) = \det(A)\det(B)$

Theorem: A matrix A is singular if and only if $\det A = 0$, in other words, A matrix A is nonsingular if and only if $\det A \neq 0$.

Exercise: Let a square matrix A be nonsingular. Show that $\det(A^{-1}) = \dfrac{1}{\det(A)}$

$$A^{-1}A = I$$
$$\det(A^{-1}A) = \det(I) = 1$$
$$\det(A^{-1}A) = \det(A^{-1})\det(A)$$
$$\det(A^{-1})\det(A) = 1$$
$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Example 4: Find the determinant of the following matrix using property of (e) and (g):

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -5 & 7 \\ 0 & -2 & 5 \end{vmatrix} \begin{matrix} \\ (-2R_1 + R_2 \to R_2) \\ (-R_1 + R_3 \to R_3) \end{matrix}$$

$$= \begin{vmatrix} 1 & 2 & -3 \\ 0 & -5 & 7 \\ 0 & -2 & 5 \end{vmatrix} \quad (-\frac{2}{5}R_2 + R_3 \to R_3)$$

$$= \begin{vmatrix} 1 & 2 & -3 \\ 0 & -5 & 7 \\ 0 & 0 & \frac{11}{5} \end{vmatrix} = 1 \bullet (-5)\frac{11}{5} = -11$$

Homework:

2.1 1,2,3,(e),(g),(f), 4, 12

2.2 1,(b),(c),6

Matlab

Example 1: Using matlab to compute the above determinant of matrix A

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>> A=[1 2 -3;2 -1 1;1 0 2]

>> det(A)

Ans=-11
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