3.1 Vector Space Definition and Examples

Definition: A **vector space** is a set V on which two operations: addition ($+$) and scalar multiplication ($\bullet$) are defined, and the following conditions are satisfied:

The operation $+$ (vector addition) must satisfy the following conditions:

Closure under addition: If $x \in V, y \in V,$ then $x + y \in V$.

(1). Commutative law: For all $x, y \in V, x + y = y + x$

(2). Associative law: $x, y, z \in V, x + (y + z) = (x + y) + z$

(3). Zero element: The set V contains a zero element, denoted by 0 (also called zero vector) such that $0 + x = x + 0 = x$

(4). Additive inverses: For any $x \in V$, there exists an element $y \in V$ (also called negative of x), such that $x + y = y + x = 0$

The operation $\bullet$ (scalar multiplication) is defined between scalars and vectors, and must satisfy the following conditions:

Closure under scalar multiplication: If $x \in V$, and α is a scalar, then the product $\alpha \bullet x \in V$

(5). Distributive law: If $x, y \in V$, and α is a scalar, then the product $\alpha \bullet (x + y) = \alpha \bullet x + \alpha \bullet y$

(6). Distributive law: If $x \in V$, and α, β is a scalar, then the product $(\alpha + \beta) \bullet x = \alpha \bullet x + \beta \bullet x$

(7). Associative law: If $x \in V$, and α, β is a scalar, then $(\alpha\beta) \bullet x = \alpha \bullet (\beta \bullet x)$

(8). Unitary law: If $x \in V$, then $1 \bullet x = x$

Example 1: Let $R^{m\times n}$ denote the set of all $m \times n$ matrices with real entries (elements). Let $A = (a_{ij})_{m\times n}, B = (b_{ij})_{m\times n} \in R^{m\times n}$, α is a real number. Vector addition is defined by $A + B = (a_{ij} + b_{ij})_{m\times n}$. Scalar multiplication is defined as $\alpha A = (\alpha a_{ij})_{m\times n}$. Prove $R^{m\times n}$ is a vector space.

Prove: We have to check Closure under addition and scalar multiplication on top of the other 8 conditions. Given any $A = (a_{ij})_{m\times n}, B = (b_{ij})_{m\times n}, C = (c_{ij})_{m\times n} \in R^{m\times n}$, α, β are real numbers.

Closure under addition: It is clear that $A + B = (a_{ij} + b_{ij})_{m\times n} \in R^{m\times n}$

(1). $A + B = (a_{ij} + b_{ij})_{m\times n} = (b_{ij} + a_{ij})_{m\times n} = B + A$

(2). $A+(B+C)=(a_{ij})_{m\times n}+(b_{ij}+c_{ij})_{m\times n}=(a_{ij}+b_{ij}+c_{ij})_{m\times n}$

$(A+B)+C=(a_{ij}+b_{ij})_{m\times n}+(c_{ij})_{m\times n}=(a_{ij}+b_{ij}+c_{ij})_{m\times n}$
$A+(B+C)=(A+B)+C$

(3). zero matrix $0=(0)_{m\times n}$ is a zero element in $R^{m\times n}$

(4). For any $A=(a_{ij})_{m\times n}\in R^{m\times n}$, there exists $B=(-a_{ij})_{m\times n}\in R^{m\times n}$, such that $A+B=B+A=0$. So there exists an additive inverse (negative)

Closure under scalar multiplication: $\alpha A=(\alpha a_{ij})_{m\times n}\in R^{m\times n}$

(5). $\alpha(A+B)=\alpha(a_{ij}+b_{ij})_{m\times n}=(\alpha a_{ij}+\alpha b_{ij})_{m\times n}=(\alpha a_{ij})_{m\times n}+(\alpha b_{ij})_{m\times n}=\alpha A+\beta B$

(6). $(\alpha+\beta)A=((\alpha+\beta)a_{ij})_{m\times n}=(\alpha a_{ij}+\beta a_{ij})_{m\times n}=(\alpha a_{ij})_{m\times n}+(\beta a_{ij})_{m\times n}=\alpha A+\beta A$

(7). $(\alpha\beta)A=((\alpha\beta)a_{ij})_{m\times n}=(\alpha(\beta a_{ij}))_{m\times n}=\alpha(\beta a_{ij})_{m\times n}=\alpha(\beta A)$

(8). $1A=(1a_{ij})_{m\times n}=(a_{ij})_{m\times n}=A$

Exercise: Let $F[a,b]$ be the set of all real valued functions that are defined on the interval [a,b]. Then given any two "vectors", $f=f(x), g=g(x)\in F[a,b]$, and any scalar α define addition and scalar multiplication as,

$(f+g)(x)=f(x)+g(x), \quad (\alpha f)(x)=\alpha f(x)$

Under these operations $F[a,b]$ is a vector space.

Example 2: Suppose that the set V is the set of positive real numbers (i.e, $x>0$) with addition and scalar multiplication defined as follows: Let $x,y\in V,\alpha$ a real number,

$x\oplus y=xy, \alpha\otimes x=x^{\alpha}$

Prove: Closure under addition: $x\oplus y=xy$. Since $x>0, y>0$, so $xy>0$, i.e., $xy\in V$,so $x\oplus y\in V$

(1). $x\oplus y=xy=yx=y\oplus x$

(2). $(x\oplus y)\oplus z=(xy)\oplus z=(xy)z=x(yz)=x\oplus(yz)=x\oplus(y\oplus z)$

(3). Scalar 1 is the zero element or zero vector since $x\oplus 1=x1=x=x1=x\oplus 1$

(4). Additive inverse (negative) of x is $\frac{1}{x}$ since $x \oplus \frac{1}{x} = x\frac{1}{x} = 1, \qquad \frac{1}{x} \oplus x = \frac{1}{x}x = 1$

Closure under multiplication: $\alpha \otimes x = x^{\alpha} > 0,$ so $\alpha \otimes x \in V$

(5). $\alpha \otimes (x \oplus y) = \alpha \otimes (xy) = (xy)^{\alpha} = x^{\alpha} y^{\alpha} = x^{\alpha} \oplus y^{\alpha} = \alpha \otimes x \oplus \alpha \otimes y$

(6). $(\alpha + \beta) \otimes x = x^{\alpha+\beta} = x^{\alpha} x^{\beta} = \alpha \otimes x \oplus \beta \otimes x$

(7). $(\alpha\beta) \otimes x = x^{\alpha\beta} = (x^{\beta})^{\alpha} = \alpha \otimes x^{\beta} = \alpha \otimes (\beta \otimes x)$

(8). $1 \otimes x = x^{1} = x$

Example 3: The set $V = R^2$ with the vector addition and scalar multiplication defined as following: Let $u = (u_1, u_2), v = (v_1, v_2) \in R^2, \alpha$ is a scalar,

$$u + v = (u_1 + v_1, u_2 + v_2), \ \alpha \otimes u = (u_1, \alpha u_2)$$

is NOT a vector space.

It's clear two closure conditions are satisfied. We just check every condition as before, you will find conditions (1)-(4) correct.

(6). Let β be a scalar.

$$(\alpha + \beta) \otimes u = (u_1, (\alpha + \beta) u_2)$$
$$\alpha \otimes u + \beta \otimes u = (u_1, \alpha u_2) + (u_1, \beta u_2) = (2u_1, (\alpha + \beta) u_2)$$

So $(\alpha + \beta) \otimes u \neq \alpha \otimes u + \beta \otimes u$

Condition (6) is not valid, thus it is not a vector space.

Example 4: The set $V = R^2$ with the vector addition and scalar multiplication defined as following: Let $u = (u_1, u_2), v = (v_1, v_2) \in R^2, \alpha$ is a scalar,

$$u + v = (u_1 + v_1, u_2 + v_2), \ \alpha \otimes u = (0, \alpha u_2)$$

is NOT a vector space.

It's clear two closure conditions are satisfied. We just check every condition as before, you will find conditions (1)-(7) correct.

(8). $1 \otimes u = (0, u_2) \neq (u_1, u_2) = u$

Condition (8) is not valid, thus it is not a vector space.

Exercise: The set $V = R^2$ with the vector addition and scalar multiplication defined as following: Let $u=(u_1,u_2), v=(v_1,v_2)\in R^2, \alpha$ is a scalar,

$$u \oplus v = (u_1+v_1, 0),\ \alpha u = (\alpha u_1, \alpha u_2)$$

is NOT a vector space.

It's clear two closure conditions are satisfied. We just check every condition as before, you will find conditions (1)-(5) correct.

(6).

$$(\alpha+\beta)(u_1,u_2) = ((\alpha+\beta)u_1, (\alpha+\beta)u_2)$$
$$\alpha u \oplus \beta u = (\alpha u_1, \alpha u_2) \oplus (\beta u_1, \beta u_2) = ((\alpha+\beta)u_1, 0)$$

$$(\alpha+\beta)u \neq \alpha u \oplus \beta u$$

Exercise: Let $V = \{(1,a) \mid a \in R\}$ with addition and scalar multiplication defined as following:

Let $u=(u_1,u_2), v=(v_1,v_2)\in R^2, \alpha$ is a scalar,

$$u \oplus v = (u_1+v_1, u_2+v_2),\ \alpha u = (\alpha u_1, \alpha u_2)$$

is NOT a vector space.

(You can easily find out it does not satisfy Closure under Addition. $(1,2)+(1,3)=(2,5)\notin V$)

Properties: Let V is a vector space and x is any element of V, then

- (i). $0x = 0$
- (ii). $x + y = 0$ implies that $y = -x$ (the additive inverse x is unique)
- (iii). $(-1)x = -x$

HW: 5, 12, 13,15.