3.3 Linear Independence

In this section, we restrict ourselves to vector spaces that can be generated from a finite set of elements. Each vector in the vector space can be built up from the elements in this generating set using only the operations of addition and scalar multiplication. The generating set is referred to as a spanning set. In particular, it is desirable to find a "minimal" spanning set. For this, we need to introduce the concepts of linearly independence and linearly dependence.

Definition: The vectors $v_1, v_2, \cdots, v_n$ are in a vector space V. For the following equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

If the above equation only has a trivial solution: $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, then vectors $v_1, v_2, \cdots, v_n$ are called **linearly independent**; otherwise, vectors $v_1, v_2, \cdots, v_n$ are called **linearly dependent**.

Example 1: Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, are they linearly dependent or independent? Since $v_1 + v_2 - v_3 = 0$, so there is a non-zero solution. They are linearly dependent. If we can't observe this relationship. We can solve as the following.

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0 \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\alpha_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\alpha_2 + \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}\alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right) \xrightarrow{-2R_1 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right) \xrightarrow{2R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-R_2 + R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $\alpha_3 = t$, we have non-zero solution when $t \neq 0$: $\begin{cases} \alpha_1 = -t \\ \alpha_2 = -t \\ \alpha_3 = t \end{cases}$, so the three vectors are linearly dependent.

Exercise: Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, are they linearly dependent or independent?

$$\alpha_1 v_1 + \alpha_2 v_2 = 0$$

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\alpha_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\alpha_2 = \begin{pmatrix} \alpha_1 + \alpha_2 \\ \alpha_2 \\ 2\alpha_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \alpha_1 = \alpha_2 = 0$$

So they are linearly independent.

Exercise: Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = 0$, are they linearly dependent or independent?

$0 \bullet v_1 + 0 \bullet v_2 + 1 \bullet v_3 = 0$

$\alpha_1 = \alpha_2 = 0, \alpha_3 = 1$ is a non-trivial solution, so they are linearly dependent.

Remark: Any vector set contains 0 vector must be linearly dependent.

Exercise: True or False for following statements:

(i). Let $v_1, v_2, \cdots, v_n$ be linearly dependent, then $v_1, v_2, \cdots, v_n, v_{n+1}$ are linearly dependent.
(ii). Let $v_1, v_2, \cdots, v_n$ be linearly independent, then $v_1, v_2, \cdots, v_{n-2}, v_{n-1}$ are linearly independent.
(Both are true statements)

Example 2: Let $v_1 = x^2 - 2x + 3, v_2 = 2x^2 + x + 8, v_3 = x^2 + 8x + 7$. v_1, v_2, v_3 are linearly independent or dependent?

Solve $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$$\Rightarrow \alpha_1(x^2 - 2x + 3) + \alpha_2(2x^2 + x + 8) + \alpha_3(x^2 + 8x + 7) = 0 \Rightarrow$$
$$\Rightarrow (\alpha_1 + 2\alpha_2 + \alpha_3)x^2 + (-2\alpha_1 + \alpha_2 + 8\alpha_3)x + (3\alpha_1 + 8\alpha_2 + 7\alpha_3) = 0$$
$$\begin{cases} \alpha_1 + 2\alpha_2 + \alpha_3 = 0 \\ -2\alpha_1 + \alpha_2 + 8\alpha_3 = 0 \\ 3\alpha_1 + 8\alpha_2 + 7\alpha_3 = 0 \end{cases}$$
$$\begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 3 & 8 & 7 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 3 & 8 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 5 & 10 \\ 0 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 5 & 10 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

The determinant of coefficient matrix is 0, so it is singular. Thus, there are non-trivial solutions to the above equation. Thus, they are linearly dependent.

Property: Let vectors $x_1, x_2, \cdots, x_k$ are in a vector space R^n. If $k > n$, then the vectors $x_1, x_2, \cdots, x_k$ will be linearly dependent.

Exercise: Let $v_1 = \begin{pmatrix}1\\0\\2\end{pmatrix}, v_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}, v_3 = \begin{pmatrix}2\\1\\3\end{pmatrix}, v_4 = \begin{pmatrix}1\\0\\8\end{pmatrix}$, are they linearly dependent or independent?

Linear dependent by the above property.

Property: Let vectors $x_1, x_2, \cdots, x_n$ are in a vector space R^n and let $v_i = (x_{1i}, x_{2i}, \cdots, x_{ni})^T$ for $i = 1, 2 \cdots, n.$ If $X = (x_1, x_2, \cdots, x_n)$, then the vectors $x_1, x_2, \cdots, x_n$ will be linearly dependent if and only if X is singular (linearly independent if and only if X is non-singular).

Exercise: Use this property to solve Example 1.

Example 3: Let $v_1 = \begin{pmatrix}1\\0\\2\end{pmatrix}, v_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}, v_3 = \begin{pmatrix}4\\1\\2\end{pmatrix}$, are they linearly dependent or independent?

$\begin{vmatrix}1 & 1 & 4\\0 & 1 & 1\\2 & 0 & 2\end{vmatrix} = \begin{vmatrix}1 & 1\\0 & 2\end{vmatrix} + 2\begin{vmatrix}1 & 4\\1 & 1\end{vmatrix} = 2 - 6 = -4 \neq 0$, so they are linearly independent.

Definition: Let $f_1, f_2, \cdots, f_n$ be elements of $C^{n-1}[a,b]$, and define **Wronskian** of $f_1, f_2, \cdots, f_n$, denoted by $W[f_1, f_2, \cdots, f_n](x)$ on $[a,b]$ by

$$W[f_1, f_2, \cdots, f_n](x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & & f_n' \\ \vdots & & & \\ f_1^{n-1} & f_2^{n-1} & \cdots & f_n^{n-1} \end{vmatrix}$$

Property: Let $f_1, f_2, \cdots, f_n$ be elements of $C^{n-1}[a,b]$. If there exists a point x_0 such that $W[f_1, f_2, \cdots, f_n](x_0) \neq 0$, then $f_1, f_2, \cdots, f_n$ are linearly independent.

Another way to solve Example 2.

$$W[v_1, v_2, v_3](x) = \begin{vmatrix} x^2 - 2x + 3 & 2x^2 + x + 8 & x^2 + 8x + 7 \\ 2x - 2 & 4x + 1 & 2x + 8 \\ 2 & 4 & 2 \end{vmatrix}$$

$$= (x^2 - 2x + 3)\begin{vmatrix} 4x+1 & 2x+8 \\ 4 & 2 \end{vmatrix} - (2x-2)\begin{vmatrix} 2x^2 + x + 8 & x^2 + 8x + 7 \\ 4 & 2 \end{vmatrix} + 2\begin{vmatrix} 2x^2 + x + 8 & x^2 + 8x + 7 \\ 4x+1 & 2x+8 \end{vmatrix} = \cdots \equiv 0$$

So, v_1, v_2, v_3, are linearly dependent.

Property: Let $v_1, v_2, \cdots, v_n$ be vectors in a vector space V, a vector $v \in span(v_1, v_2, \cdots, v_n)$ can be written uniquely as a linear combination of $v_1, v_2, \cdots, v_n$ if and only if $v_1, v_2, \cdots, v_n$ are linearly independent.

HW: 2(a),(b),4, 5,6,8,10