3.4 Basis and Dimension

In last section, we showed that a spanning set for a vector space is minimal if its elements are linearly independent. The elements of a minimal spanning set form the basic building blocks for the whole vector space and, consequently, we say that they form a "basis" for the vector space.

Definition: Suppose $S=\{v_1,v_2,\cdots,v_n\}$ is a set of vectors in a vector space V, then S is called a **basis** (plural **bases**) of V, if S satisfies the following two conditions:

(1). $v_1,v_2,\cdots,v_n$ are linearly independent,

(2). $span(S)=span(v_1,v_2,\cdots,v_n)=V$.

Example 1: The "standard basis" (the most natural way, the simplest way to represent vectors in vector space) for R^n is $e_1=(1,0,\cdots,0)^T, e_2=(0,1,\cdots,0)^T,\cdots,e_n=(0,0,\cdots,1)^T$, the "standard basis" for $M_{2\times2}$ is $E_{11}=\begin{pmatrix}1&0\\0&0\end{pmatrix}, E_{12}=\begin{pmatrix}0&1\\0&0\end{pmatrix}, E_{21}=\begin{pmatrix}0&0\\1&0\end{pmatrix}, E_{22}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$, the "standard basis" for P_n is $1,x,x^2,\cdots,x^{n-1}$.

Definition: If $S=\{v_1,v_2,\cdots,v_n\}$ is a basis of a non-zero vector space V, then we call V a **finite dimensional vector space**, and we say that the **dimension** of V, denoted by $\dim(V)$, is n. (the number of elements in S **).** If V is not a finite dimensional vector space**,** then we call it an **infinite dimensional** vector space**.** The dimension for zero space is 0.

Remark 1: If W is a subspace of vector space V, then $\dim(W)\le\dim(V)$.

Remark 2: Given $W=\{v_1,v_2,\cdots,v_m\}$ is a subspace of vector space V, and $\dim(V)=n$. If $m>n$, then $W=\{v_1,v_2,\cdots,v_m\}$ is linearly dependent.

Remark 3: If $W=\{v_1,v_2,\cdots,v_m\}$ is a **basis** of vector space V, and $\dim(V)=n$, then $m=n.$.

Example 2: $\dim(R^n)=n, \dim(M_{m\times n})=mn, \dim(P_n)=n, \ {}^*\dim(P)=\infty.$

The polynomial set, P include any P_n $(P\supseteq P_n)$, so $\dim(P)\ge\dim(P_n)=n$, let $n\to\infty$, we can see $\dim(P)=\infty.$

Properties**:** Given $\dim(V)=n\ (n>0)$, $S=\{v_1,v_2,\cdots,v_n\}\subseteq V$, then

(i). If $v_1,v_2,\cdots,v_n$ are linearly independent, then $S=\{v_1,v_2,\cdots,v_n\}$ is a basis of V.

(ii). If $span(v_1,v_2,\cdots,v_n)=V$, then $S=\{v_1,v_2,\cdots,v_n\}$ is a basis of V.

(iii). If $m<n$, then $span\{v_1,v_2,\cdots,v_m\}\subset(\neq)V$.

(iv). If $W=\{v_1,v_2,\cdots,v_m\}\subseteq V$ is linearly independent, but is not a base for V then can be enlarged to a basis for V by adding in certain vectors from V.

(iv). If $span\{v_1,v_2,\cdots,v_m\}=V, \quad m>n$, then $W=\{v_1,v_2,\cdots,v_m\}$ is linearly dependent, and can be reduced to a basis for V by removing certain vectors from W.

(v). Given $S=\{v_1,v_2,\cdots,v_n\}\subseteq R^n$, if the determinant of matrix formed by $v_1,v_2,\cdots,v_n$ is not zero, then $S=\{v_1,v_2,\cdots,v_n\}$ is a basis for R^n.

Example 3: Determine whether $\left\{\begin{pmatrix}1\\0\\-2\end{pmatrix},\begin{pmatrix}-1\\3\\2\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right\}$ is a basis for R^3.

$$\begin{vmatrix}1 & -1 & 2\\0 & 3 & 1\\-2 & 2 & 1\end{vmatrix}=\begin{vmatrix}3 & 1\\2 & 1\end{vmatrix}-2\begin{vmatrix}-1 & 2\\3 & 1\end{vmatrix}=1-2(-7)=15\neq 0$$

So, it is a basis for R^3.

Exercise 1: Determine whether $\left\{\begin{pmatrix}1\\0\\-2\end{pmatrix},\begin{pmatrix}-1\\3\\2\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}3\\5\\4\end{pmatrix}\right\}$ is linearly independent, a basis for R^3.

Neither linearly independent nor a basis for R^3.

Exercise 2: Let $W=\{x_1,x_2\}$, where $x_1=\begin{pmatrix}1\\0\\-2\end{pmatrix}, x_2=\begin{pmatrix}-1\\3\\2\end{pmatrix}$.

(a). Determine whether W is linearly independent, a basis for R^3.

(b). Let x_3 be a third vector in R^3 and set $X=\{x_1,x_2,x_3\}$ what condition(s) would X have to satisfy in order for x_1,x_2,x_3 to form a basis for R^3.

(c). Find a vector x_3 that will extend the set $\{x_1,x_2\}$ to a basis for R^3.

(a).

$$\begin{pmatrix}1\\0\\-2\end{pmatrix}\alpha_1+\begin{pmatrix}-1\\3\\2\end{pmatrix}\alpha_2=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}\alpha_1-\alpha_2\\3\alpha_2\\-2\alpha_1+2\alpha_2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\begin{cases}\alpha_1-\alpha_2=0\\3\alpha_2=0\\-2\alpha_1+2\alpha_2=0\end{cases}\Rightarrow\begin{cases}\alpha_1=0\\\alpha_2=0\end{cases}$$

$\left\{\begin{pmatrix}1\\0\\-2\end{pmatrix},\begin{pmatrix}-1\\3\\2\end{pmatrix}\right\}$ is linearly independent, but not a basis for R^3.

(b). $|X| \neq 0$

(c). Let $x_3 = (a,b,c)^T$.

$$|X| = \begin{vmatrix}1 & -1 & a\\0 & 3 & b\\-2 & 2 & c\end{vmatrix} = \begin{vmatrix}3 & b\\2 & c\end{vmatrix} - 2\begin{vmatrix}-1 & a\\3 & b\end{vmatrix} = 3c - 2b - 2(-b-3a) = 3c + 6b$$

As long as $|X| \neq 0$, in other words, $3c + 6b \neq 0$, we have x_1, x_2, x_3 are independent, x_1, x_2, x_3 so form a basis for R^3. Let $a = 0, b = 1, c = 0$, so $x_3 = (0,1,0)^T$.

This will guarantee x_1, x_2, x_3 form a basis for R^3

Exercise 3: Given $W = \left\{\begin{pmatrix}1\\0\\-2\end{pmatrix},\begin{pmatrix}2\\3\\2\end{pmatrix},\begin{pmatrix}4\\3\\-2\end{pmatrix}\right\}$

(a). Determine whether W is linearly independent, a basis for R^3.

(b). Find the dimension of $span(W)$.

(a).

$$\begin{vmatrix}1 & 2 & 4\\0 & 3 & 3\\-2 & 2 & -2\end{vmatrix} = \begin{vmatrix}3 & 3\\2 & -2\end{vmatrix} - 2\begin{vmatrix}2 & 4\\3 & 3\end{vmatrix} = 0$$

W is linearly dependent, so it is not a basis for R^3.

(b).

Since W is linearly dependent, so the dimension of $span(W)$ can't be 3.

Also because $span(W) \subseteq R^3$, so $\dim(span(W)) \leq \dim(R^3) = 3$, thus $\dim(span(W)) \leq 2$. So we need to determine whether the dimension of $span(W)$ is 1 or 2. We just check whether any two vectors are linearly dependent, if any two of them are linearly independent, then we can stop checking and the dimension is 2.

$$\begin{pmatrix}1\\0\\-2\end{pmatrix}\alpha_1+\begin{pmatrix}2\\3\\2\end{pmatrix}\alpha_2=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}\alpha_1+2\alpha_2\\3\alpha_2\\-2\alpha_1+2\alpha_2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\begin{cases}\alpha_1+2\alpha_2=0\\3\alpha_2=0\\-2\alpha_1+2\alpha_2=0\end{cases}\Rightarrow\begin{cases}\alpha_1=0\\\alpha_2=0\end{cases}$$

$\left\{\begin{pmatrix}1\\0\\-2\end{pmatrix},\begin{pmatrix}2\\3\\2\end{pmatrix}\right\}$ is linearly independent, so the dimension of $\dim(span(W))\geq 2$ thus

$\dim(span(W))=2$ since we already know that $\dim(span(W))\leq 2$.

Example 4: The vectors: $x_0=\begin{pmatrix}2\\0\\-4\end{pmatrix}, x_1=\begin{pmatrix}1\\0\\-2\end{pmatrix}, x_2=\begin{pmatrix}2\\3\\2\end{pmatrix}, x_3=\begin{pmatrix}4\\3\\-2\end{pmatrix}, x_4=\begin{pmatrix}1\\3\\2\end{pmatrix}, x_5=\begin{pmatrix}-2\\-3\\-2\end{pmatrix}$

span R^3. Choose vectors from $x_0, x_1, x_2, x_3, x_4, x_5$ to form a basis for R^3.

Since it is clear that

$x_0=2x_1, x_2=-x_5$, x_0, x_1 are related, x_2, x_5 are related. So remove x_0, x_5 from the candidate (of course, you can remove x_1, x_2) for the basis. Now, we choose 3 independent vectors from x_1, x_2, x_3, x_4, remember as long as the determinant of the matrix formed by three vectors are not zero, then the three vectors are independent, thus they form a basis for R^3. First try x_1, x_2, x_3.

$\begin{vmatrix}1&2&4\\0&3&3\\-2&2&-2\end{vmatrix}=0$, so x_1, x_2, x_3 are linearly dependent, thus they do not form a basis. Following we try x_1, x_2, x_4.

$\begin{vmatrix}1&2&1\\0&3&3\\-2&2&2\end{vmatrix}=-6\neq 0$, so x_1, x_2, x_4 are linearly independent, thus they form a basis.

Exercise 4: Given $A=\begin{pmatrix}1&2&0&3\\1&0&2&1\\-1&0&-2&-1\\2&4&0&6\end{pmatrix}$. Find the null space of A and the dimension of the null space.

First, we recall that to find the null space of a mtrix, we need to solve the following system:

$$\begin{pmatrix}1&2&0&3\\1&0&2&1\\-1&0&-2&-1\\2&4&0&6\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & 1 & 0 \\ -1 & 0 & -2 & -1 & 0 \\ 2 & 4 & 0 & 6 & 0 \end{array}\right) \xrightarrow[-2R_1+R_4 \to R_4]{\substack{-R_1+R_2 \to R_2 \\ R_1+R_3 \to R_3}} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & -2 & 2 & -2 & 0 \\ 0 & 2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2+R_3 \to R_3} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & -2 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2+R_1 \to R_1} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & -2 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-\frac{1}{2}R_2 \to R_2} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 = -2x_3 - x_4 \\ x_2 = x_3 - x_4 \end{cases}$$

Let $x_3 = t, x_4 = s$, we have

$$\begin{cases} x_1 = -2t - s \\ x_2 = t - s \\ x_3 = t \\ x_4 = s \end{cases} \to \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2t - s \\ t - s \\ t \\ s \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} s$$

So that we can see that the null space is the space that is the set of all vectors that are a linear combination of

$\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$. So the null space of A is spanned by these two vectors. We can easily check that the two vectors are linearly independent. Thus they form a basis of the null space, and from this we know the dimension of the null space is 2.

HW: 3. 4,5,8,10,13,14.