3.5 Change of Basis

The standard basis for any vector space is generally the easiest to work with, but unfortunately there are times when we need to work with other bases. In this section we discuss the problem of switching from one basis to another one.

Definition 1: Suppose that $S = \{s_1, s_2, \ldots, s_n\}$ is a basis for a vector space V and that x is any vector from V. Since x is a vector in , it can be expressed as a linear combination of the vectors from S as follows,

$$x = c_1 s + c_2 s_2 + \cdots + c_n s_n$$

The scalars $c_1, c_2, \ldots, c_n$, are called the coordinates. $(c_1, c_2, \ldots, c_n)^T$ is called the **coordinate vector** of x relative to the basis S, denoted by $[x]_S$.

Example 1: Determine the coordinate vector of $x = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ relative to the following bases.

(a). The standard basis for $R^2, E = \{e_1, e_2\}$

(b). The basis $S = \{s_1, s_2\}$, $s_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, s_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Solution: (a). $x = \begin{pmatrix} 2 \\ -4 \end{pmatrix} = 2e_1 - 4e_2$. The coordinate vector relative to the standard basis is $[x]_E = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

(b). Let $x = as_1 + bs_2$ since $\{s_1, s_2\}$ is a basis, so we have

$$\because x = \begin{pmatrix} 1 \\ -2 \end{pmatrix} a + \begin{pmatrix} 1 \\ 1 \end{pmatrix} b = \begin{pmatrix} a+b \\ -2a+b \end{pmatrix}$$

$$x = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

$$\therefore \begin{pmatrix} a+b \\ -2a+b \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

Solving this system, we have $a = 2, b = 0$

Thus, the coordinate vector relative to $S = \{s_1, s_2\}$ is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, or $[x]_s = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Question: Given a vector coordinates with respect to basis $S = \{s_1, s_2\}$ in R^2, where $s_1 = \begin{pmatrix} s_{11} \\ s_{21} \end{pmatrix}, s_2 = \begin{pmatrix} s_{12} \\ s_{22} \end{pmatrix}$, $[x]_S = (\mathrm{a}, \mathrm{b})$, in other words, $x = \mathrm{a}s_1 + bs_2$. Write down the coordinates of vector x with respect to standard basis $E = \{e_1, e_2\}$.

$$x = as_1 + bs_2 = a(s_{11}e_1 + s_{21}e_2) + b(s_{12}e_1 + s_{22}e_2) = (as_{11} + bs_{12})e_1 + (as_{21} + bs_{22})e_2$$

$$\therefore [x]_E = \begin{pmatrix} as_{11} + bs_{12} \\ as_{21} + bs_{22} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

Or

$$[x]_E = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} [x]_s$$

We call $\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$ is the transition matrix from the ordered basis: $S = \{s_1, s_2\}$ to standard basis: $E = \{e_1, e_2\}$.

Remark: Since $[x]_E = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} [x]_s$, we have $[u]_s = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}^{-1} [u]_E$. We call $\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}^{-1}$ is the transition matrix from standard basis: $E = \{e_1, e_2\}$ to the ordered basis: $= \{s_1, s_2\}$.

Exercise (review): If $ad - bc \neq 0$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Solution: $\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} ad - bc & bd - bd \\ -ac + ac & ad - bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Remark: It's good idea if you can remember this result even it is not required. Well, you need to know how to get it by row operations.

Example 2: Given the basis $S = \{s_1, s_2\}$, $s_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, s_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. A point x with the coordinate vector relative to S is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, or $[x]_s = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Find its coordinate vector relative to standard basis: $E = \{e_1, e_2\}$.

$$[x]_E = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} [x]_s = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

Example 3: Given a basis $U = \{u_1, u_2\}$, $u_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, u_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$. A point x with the coordinate vector relative to standard basis: $E = \{e_1, e_2\}$ is $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$. Find its coordinate vector relative to basis: $U = \{u_1, u_2\}$, or $[x]_U$.

Solution: $[x]_U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}^{-1} [x]_E = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \frac{1}{5-6} \begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -18 \\ 10 \end{pmatrix}$

Since $[x]_E = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}[x]_s$ and $[x]_E = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}[x]_U$, so we have

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}[x]_U = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}[x]_s$$

or
$$[x]_U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}^{-1}\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}[x]_s$$

Thus, we call $\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}^{-1}\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$ is the transition matrix from ordered basis $S = \{s_1, s_2\}$ to ordered basis $U = \{u_1, u_2\}$.

Exercise 1: Given a basis $S = \{s_1, s_2\}$, $s_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, s_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and another basis $U = \{u_1, u_2\}$, $u_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, u_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$. A point x with the coordinate vector relative to S is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, or $[x]_s = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Find its coordinate vector relative to basis: $U = \{u_1, u_2\}$, or $[x]_U$.

$$[x]_U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}^{-1}\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}[x]_s = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -18 \\ 10 \end{pmatrix}$$

Method 2 (I prefer since I do not need to remember the formula).

Solution: A point x with the coordinate vector relative to S is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, so

$$x = 2s_1 + 0s_2 = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

Suppose x coordinate vector relative to basis: $U = \{u_1, u_2\}$ is $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, so

$$x = \alpha u_1 + \beta u_2 = \alpha\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \beta\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} \alpha + 2\beta \\ 3\alpha + 5\beta \end{pmatrix}$$

Thus, we have

$$\begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} \alpha + 2\beta \\ 3\alpha + 5\beta \end{pmatrix} \rightarrow \begin{cases} \alpha + 2\beta = 2 \\ 3\alpha + 5\beta = -4 \end{cases} \rightarrow \begin{cases} \alpha = -18 \\ \beta = 10 \end{cases}$$

So the coordinates relative to basis U is $\begin{pmatrix} -18 \\ 10 \end{pmatrix}$

Example 4*: Given $s_1 = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, s_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, S = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}$.

Find vectors u_1, u_2 so that S will be the transition matrix from $[s_1, s_2]$ to $[u_1, u_2]$.

$$S = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}^{-1} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 5 \end{pmatrix}$$

$$u_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

Exercise: 1, 3,4.