3.6 Row Space and Column Space

In this section, we want to take a look at some important subspaces that are associated with matrices.

Definition 1: If A is an $m \times n$ matrix, the subspace of $R^{1\times n}$ spanned by the row vectors of A is called the **row space** of A. The subspace of R^m spanned by the column vectors of A is called the **column space** of A.

Example 1: Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & 2 \\ 4 & 0 & 8 \end{pmatrix}$.

Row vectors are $r_1 = (1,2,3), r_2 = (2,4,6), r_3 = (1,0,2), r_4 = (4,0,8)$.

Let R be the row space of A, the space spanned by r_1, r_2, r_3, r_4, or $R = span(r_1, r_2, r_3, r_4)$

Column vectors are $c_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 4 \end{pmatrix}, c_2 = \begin{pmatrix} 2 \\ 4 \\ 0 \\ 0 \end{pmatrix}, c_3 = \begin{pmatrix} 3 \\ 6 \\ 2 \\ 8 \end{pmatrix}$.

Let C be the column space of A, the space spanned by c_1, c_2, c_3, or $C = span(c_1, c_2, c_3)$.

Property 1: Suppose that the matrix D is in row-echelon form. The row vectors containing leading 1's (so the non-zero row vectors) will form a basis for the row space of D. The column vectors that contain the leading 1's from the column vectors will form a basis for the column space of D.

Example 2: Given $D = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, the basis for row space of D is $\{(1 \quad 2 \quad 3), (0 \quad 1 \quad \frac{1}{2})\}$, the basis for column space of D is $\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}\}$.

How to find the basis of row space R and column space C in this example?

Let us use row operations

$1).R_i \leftrightarrow R_j$
$2).R_i \to aR_i (a \neq 0)$
$3).aR_i + R_j \to R_j$

to reduce the matrix to row-chelon form.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & 2 \\ 4 & 0 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & -8 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to D = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Property 2: Two row equivalent matrices have the same row space.

Property 3: Suppose that A and D are two row equivalent matrices, then a set of column vectors from A will be a basis for the column space of A if and only if the corresponding columns from D will form a basis for the column space of D.

By the above properties, we have

Row space of A is clearly the spanned space: $span((1 \quad 2 \quad 3),(0 \quad 1 \quad \frac{1}{2}))$

Column space of A is clearly the spanned space: $span(\begin{pmatrix} 1 \\ 2 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \\ 0 \end{pmatrix})$

Property 4: The dimension of the row space of A is the same as the dimension of the column space of A.

Definition 2: We call the dimension of the row space of A (or dimension of the column space of A), the rank of A, denoted by $rank(A)$

Definition 3: The dimension of the null space of A is called the nullity of A, denoted by $nullity(A)$.

Property 5: Given A is an $m \times n$ matrix, then $rank(A) + nullity(A) = n$

Example 2: Find a basis for the null space, row space and column space of the following matrix A. Determine the nullity and rank of this matrix.

$$A=\begin{pmatrix} 1 & -1 & 0 & 2 \\ 1 & 0 & 1 & 2 \\ -2 & 0 & -2 & -4 \\ 2 & -2 & 0 & 2 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 2 & 0 \\ -2 & 0 & -2 & -4 & 0 \\ 2 & -2 & 0 & 2 & 0 \end{array}\right) \xrightarrow[\substack{2R_1+R_3\to R_3 \\ -2R_1+R_4\to R_4}]{-R_1+R_2\to R_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right) \xrightarrow[-\frac{1}{2}R_4\to R_4]{2R_2+R_3\to R_3} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R_3\leftrightarrow R_4]{-2R_4+R_1\to R_1} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \begin{cases} x_1=-x_3 \\ x_2=-x_3 \\ x_4=0 \end{cases}$$

Let $x_3=t$, we have

$$\begin{cases} x_1=-t \\ x_2=-t \\ x_3=t \\ x_4=0 \end{cases} \to \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -t \\ -t \\ t \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} t$$

so the null space is spanned by 1 vector: $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$, thus, the basis of null space is $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$, $nullity(A)=1$, so $rank(A)=4-1=3$.

row space of A, $span((1\ \ 0\ \ 1\ \ 0),(0\ \ 1\ \ 1\ \ 0),(0\ \ 0\ \ 0\ \ 1))$

column space of A, $span(\begin{pmatrix} 1 \\ 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ -4 \\ 2 \end{pmatrix})$

A linear system $Ax=b$, can be written as $\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} x_2 + \cdots + \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} x_n = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$.

It's clear that if there is a solution, that means $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ can be written as the combination of $\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \cdots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$

Thus, b is in the column space of A.

Property 6: The system of linear equations $Ax=b$ will be consistent (i.e. have at least one solution) if and only b is in the column space of A.

Property 7: Given A is an $n \times n$ matrix, then the following statements are equivalent.

a. A is invertible
b. $\det(A) \neq 0$
c. The only solution to the system $Ax = 0$ is the trivial solution
d. The null space of A is {0}, i.e. just the zero vector.
e. $nullity(A) = 0$
f. $rank(A) = n$
g. $Ax = b$ has exactly one solution for every $n \times 1$ matrix b
h. $Ax = b$ is consistent for every $n \times 1$ matrix b
i. The row vectors of A form a basis for $R^{1\times n}$
j. The columns vectors of A form a basis for R^n

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