4.1 Definition and Examples

Linear Mappings from one vector space to another play an important role in mathematics.

Definition 1: A mapping L from a vector space V into a vector space W is said to be a **linear transformation** or **linear operator** if

(1) $$L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2)$$

for all $v_1, v_2 \in V$ and for all scalars α and β.

Condition (1) is equivalent to the following conditions:

(2) $$L(v_1 + v_2) = L(v_1) + L(v_2)$$

(3) $$L(\alpha v) = \alpha L(v)$$

We can easily check the above conclusion. Thus, L is a linear operator if and only L satisfies (2) and (3).

Notation: A mapping L from a vector space V into a vector space W will be denoted

$$L : V \to W$$

When the arrow notation is used, it will be assumed that V and W represent vector spaces.

Example 1: Let L be the operator define by

$$L(x) = 2x$$

For each $x \in R^2$.

Since $L(x+y) = 2(x+y) = 2x + 2y = L(x) + L(y)$, and $L(\alpha x) = 2\alpha x = \alpha 2x = \alpha L(x)$

Thus L is a linear operator.

Example 2: Let M be the operator define by

$$M(x) = ({x_1}^2 + {x_2}^2 + {x_3}^2)^{\frac{1}{2}}$$

For each $x \in R^3$.

Since

$$M(\alpha x) = ((\alpha x_1)^2 + (\alpha x_2)^2 + (\alpha x_3)^2)^{\frac{1}{2}} = [(\alpha^2)({x_1}^2 + {x_2}^2 + {x_3}^2)]^{\frac{1}{2}}$$
$$= |\alpha| ({x_1}^2 + {x_2}^2 + {x_3}^2)^{\frac{1}{2}} = |\alpha| M(x) \neq \alpha M(x)$$

M is not linear operator.

Let A is any $m \times n$ matrix, the following operator:

$$L_A(x) = Ax$$

from $R^n \to R^m$ for each $x \in R^n$ is a linear operator since

$$L_A(\alpha x + \beta y) = A(\alpha x + \beta y) = A(\alpha x) + A(\beta y) = \alpha Ax + \beta Ay = \alpha L_A(x) + \beta L_A(y)$$

Thus, we can think of each $m \times n$ matrix A as a linear operator from $R^n \to R^m$.

Property: If L is a linear operator mapping from a vector space V into a vector space W, then

(a). $L(0_V) = 0_W$, where 0_V and 0_W are the zero vectors in V and W respectively.

(b). $L(-v) = -L(v)$ for any $v \in V$.

(c). If $v_1, v_2, \cdots, v_n$ are elements of V and $\alpha_1, \alpha_2, \cdots, \alpha_n$ are scalars, then

$$L(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = \alpha_1 L(v_1) + \alpha_2 L(v_2) + \cdots + \alpha_n L(v_n)$$

Example 3: If V is a vector space, then the identity operator I is defined by

$$I(v) = v$$

for any $v \in V$.

It's a linear operator since $I(\alpha v + \beta y) = \alpha v + \beta y = \alpha I(v) + \beta I(y)$

Example 4: Let D be the operator mapping $C^1[a,b]$ into $C[a,b]$ defined by

$$D(f) = f' \text{ (the derivative of } f\text{)}$$

Clearly, D is a linear operator since $D(\alpha f + \beta g) = \alpha f' + \beta g' = \alpha D(f) + \beta D(g)$

Definition 2: Let $L : V \to W$ be a linear transformation (operator). The **kernel** of L, denoted by $\ker(L)$, is defined by

$$\ker(L) = \{v \in V, L(v) = 0_W\}$$

Definition 3: Let $L : V \to W$ be a linear transformation (operator) and let S be a subspace of V. The **image** of S is defined by

$$L(S) = \{w \in W \mid w = L(v) \text{ for some } v \in S\}$$

The image of the entire vector space, $L(V)$, is called the range of L.

It can easily verified that $\ker(L)$ and $L(S)$ are subspaces.

Example 5: D is a linear operator defined in Example 4. Find $\ker(D)$.

$\ker(D) = c$ (c is a constant number)

Example 6: Let L be the linear operator from $R^3 \to R^3$ defined by

$$L(x) = \begin{pmatrix} x_1 + 2x_3 \\ 0 \\ 2x_1 + 4x_2 \end{pmatrix}$$

Find $\ker(L)$.

Find $\ker(L)$ first. Let $x \in \ker(L)$, we have

$$\begin{pmatrix} x_1 + 2x_3 \\ 0 \\ 2x_1 + 4x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = -2x_3 \\ x_2 = x_3 \end{cases} \text{ Let } x_3 = t, \text{ we have}$$

$$\begin{cases} x_1 = -2t \\ x_2 = t \\ x_3 = t \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} t$$

$$\ker(L) = span(\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix})$$

HW: 1, 3, 5,7,9,16, 19.