4.3 Similarity

Definition 1: Let A and B be $n\times n$ square matrices. B is said to be similar to A if there exists a nonsingular matrix S such that $B=S^{-1}AS$.

Example 1: Let $A=\begin{pmatrix}2&0\\1&1\end{pmatrix}$, $B=\begin{pmatrix}2&-1\\0&1\end{pmatrix}$, $S=\begin{pmatrix}1&-1\\1&1\end{pmatrix}$. Check $B=S^{-1}AS$, so B is similar to A.

Solution: $S^{-1}AS=\begin{pmatrix}1&-1\\1&1\end{pmatrix}^{-1}\begin{pmatrix}2&0\\1&1\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix}=\frac{1}{1-(-1)}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}2&0\\1&1\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix}=\begin{pmatrix}2&-1\\0&1\end{pmatrix}=B$

Thus, B is similar to A.

Property 1: Similarity is **reflexive**. A square matrix A is similar to itself, since $A=I^{-1}AI$.

Property 2: Similarity is **symmetric**. If B is said to be similar to A, then A is said to be similar to B. Since B is said to be similar to A, there is a nonsingular matrix S, such that $B=S^{-1}AS$. Thus, we have $A=SBS^{-1}$. Let $R=S^{-1}, R^{-1}=(S^{-1})^{-1}=S$, so $A=R^{-1}BR$, thus A is similar to B.

Property 3: Similarity is **transitive**. If A is similar to B, and B is similar to C, then A is similar to C.

Prove: Since A is similar to B, there is a nonsingular matrix S, such that $A=S^{-1}BS$, and since B is similar to C, there is a nonsingular matrix R, such that $B=R^{-1}CR$. Thus we have
$A=S^{-1}BS=S^{-1}R^{-1}CRS=(RS)^{-1}C(RS)$ since $S^{-1}R^{-1}=(RS)^{-1}$

A is similar to C.

Property 4: if A and B are similar, then $\det(A)=\det(B)$

Prove. Since A and B are similar, there is a nonsingular matrix S, such that
$A=S^{-1}BS\rightarrow\det(A)=\det(S^{-1}BS)=\det(S^{-1})\det(B)\det(S)=\det(B)$ since $\det(S^{-1})=\frac{1}{\det(S)}$

Similarity is an equivalence relation that separates the set of all n-square matrices into equivalent classes. All matrices similar to a given matrix are similar to each other. What's more? Any matrix similar to a given matrix represents the same linear transformation as the given matrix, but as referred to a different coordinate system (or basis). Thus, any two matrices that are similar to each other represent the same linear transformation. The concept of similarity is thus intricately connected to the concept of a change in basis, a change in coordinate system. Changing the basis for a linear transformation produces similar matrices.

Changes in the expression of a linear transformation due to a change in basis.

Let $L(x)=Ax$ be a linear transformation expressed with respect to the standard basis $E=\{e_1,e_2,\cdots,e_n\}$. What is the expression for this same transformation when expressed with respect to any basis

$S = \{s_1, s_2, \cdots, s_n\}$? The transformation is $[L(x)]_S = B[x]_S$ with respect to basis S. What's the relationship of A and B?

Review: $[x]_E = S[x]_S$ *i.e.* $x = S[x]_S$ (for convenience, we write $[x]_E = x$).

$\because x = S[x]_S$, $L(x) = S[L(x)]_S$ and $L(x) = Ax$
$\therefore S[L(x)]_S = AS[x]_S$
$\therefore [L(x)]_S = S^{-1}AS[x]_S$
$\because [L(x)]_S = B[x]_S$
$\therefore B = S^{-1}AS$

Example 1: Let L be the linear operator mapping R^3 into R^3 defined by $L(x) = Ax$, where $A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix}$.

Thus, the matrix A represents L with respect to standard basis. Find the matrix representing L with respect to $S = \{s_1, s_2, s_3\}$, where $s_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, s_2 = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}, s_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Solution:

$$S = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -1 & 1 \\ -1 & 2 & 0 \end{pmatrix} \xrightarrow{\text{through row operations working on } (S|I)\text{, we will get}} S^{-1} = \begin{pmatrix} -2 & 2 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$B = S^{-1}AS = \begin{pmatrix} -2 & 2 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ 1 & -1 & 1 \\ -1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} -8 & 14 & -7 \\ -3 & 6 & -2 \\ 3 & -3 & 6 \end{pmatrix}$$

HW: 1, 5(a)(b), 6(a), 11,12,13.14