5.1 The Scalar Product in R^n

Definition 1: Let $x=(x_1,x_2,\cdots,x_n)$ and $y=(y_1,y_2,\cdots,y_n)$, then the **scalar product** of x and y are defined

$$x^T y = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

The Scalar Product in R^2 and R^3

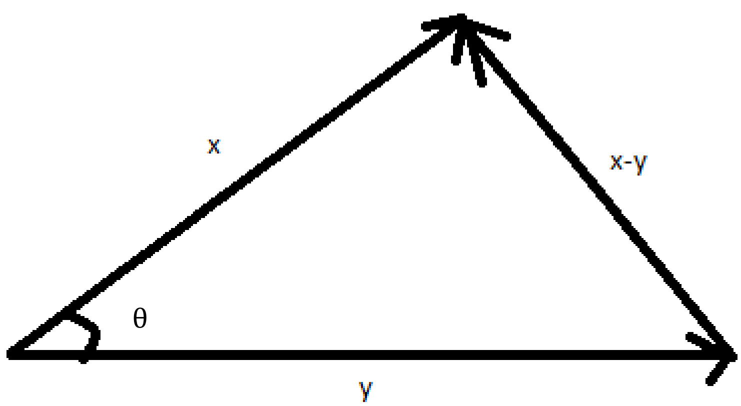
Definition 2: Given a vector x in either R^2 or R^3, its Euclidean length an de defined in terms of the scalar product $\sqrt{x^T x}$, and denoted by $\|x\|=\sqrt{x^T x}$.

First, we would like to derive some properties when the vector spaces are either R^2 or R^3 which we are familiar with, then we extend those properties to R^n.

Definition 2: Let x and y be vectors in either R^2 or R^3. The distance between x and y is defined $\|x-y\|$.

Property 1: If x and y are non-zero vectors in either R^2 or R^3 and θ is the angle between them, then

$$x^T y = \|x\| \cdot \|y\| \cos(\theta)$$



Proof: We will prove the result for R^3. The proof for R^2 is similar. By the law of cosines,

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos\theta$$

$\downarrow$

$$\|x\| \cdot \|y\| \cos\theta$$

$$= \frac{1}{2}(\|x\|^2 + \|y\|^2 - \|x-y\|^2)$$

$$= \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 - ((x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2))$$

$$= \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 - (x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 - 2x_1y_1 - 2x_2y_2 - 2x_3y_3))$$

$= x_1y_1 + x_2y_2 + x_3y_3$
$= x^T y$

Cauchy-Schwarz Inequality: If x and y be vectors in either R^2 or R^3, then

$$|x^T y| \le \|x\| \cdot \|y\|$$

Since $|\cos(\theta)| \le 1$, $|x^T y| = \|x\| \cdot \|y\| |\cos(\theta)| \le \|x\| \cdot \|y\|$

Definition 3: The vectors x and y are non-zero vectors in either R^2 or R^3 are called **orthogonal** if $x^T y = 0$.

Example 1: Prove vectors $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, y = \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix}$ are orthogonal.

Prove: $x^T y = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix} = -1 - 8 + 9 = 0$, so they are orthogonal.

Scalar and Vector projections

Excise: From Example 1, $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, y = \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix}$ are orthogonal. Please verify that $\|x\|^2 + \|y\|^2 = \|x + y\|^2$.

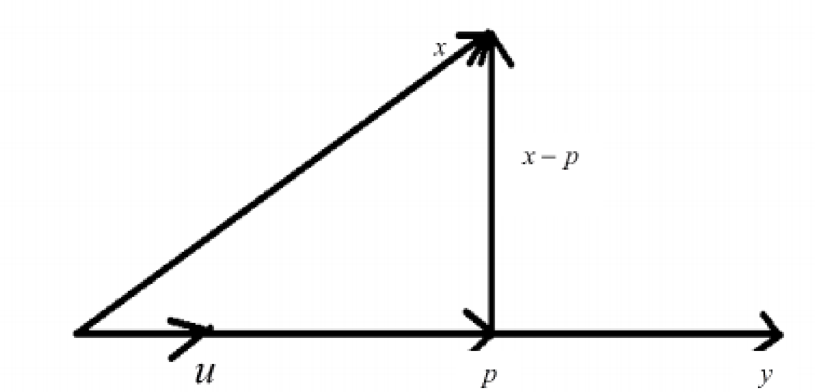
$$\|x + y\|^2 = (1 + (-1))^2 + (2 - 4)^2 + (3 + 3)^2 = 40$$

$\|x\|^2 + \|y\|^2 = 1 + 4 + 9 + 1 + 16 + 9 = 40$
Thus, $\|x + y\|^2 = \|x\|^2 + \|y\|^2$

Scalar and Vector projections in either R^2 or R^3.

The scalar product can be used to find the component of one vector in the direction of another. Let x and y be nonzero vectors in either R^2 or R^3.

Let $u = \frac{1}{\|y\|} y$ a unit vector in the direction of y. We wish to find out vector projection x onto y, vector p. We have $x - p$ orthogonal to p. Let $p = \alpha u = \alpha \frac{1}{\|y\|} y$. If we can find α, then find p.



$$(x-p)^T y = 0 \Rightarrow x^T y - p^T y = 0 \Rightarrow x^T y - \alpha \frac{y^T}{\|y\|} y = 0 \Rightarrow x^T y - \alpha \frac{\|y\|^2}{\|y\|} = 0 \Rightarrow \alpha = \frac{x^T y}{\|y\|}$$

$$p = \frac{x^T y}{\|y\|} \frac{1}{\|y\|} y = \frac{x^T y}{\|y\|^2} y$$

Scalar projection x onto y is $\alpha = \frac{x^T y}{\|y\|}$

Vector projection x onto y is $p = \frac{x^T y}{\|y\|^2} y$

Example 2: Find the distance from the point $A=(1,2,1)$ to the plane $4x+2y-z=1$.

Solution: Find a point B in plane $4x+2y-z=1$. Let $x=0, y=0 \Rightarrow z=-1$, so point $B=(0,0,-1)$ is in the given plane. The normal vector $\vec{n}$ of the plane $4x+2y-z=1$ is $\vec{n}=(4,2,-1)^T$. The distance from the point $A=(1,2,1)$ to the plane $4x+2y-z=1$ is the absolute of the scalar projection of $\overrightarrow{AB}=(-1,-2,-2)^T$ onto $\vec{n}$.

$$\alpha = \frac{\overrightarrow{AB}^T \vec{n}}{\|\vec{n}\|} = \frac{(-1,-2,-2)(4,2,-1)^T}{\sqrt{16+4+1}} = \frac{-4-4+2}{\sqrt{21}} = \frac{-6}{\sqrt{21}}$$

Thus the distance from the point $A=(1,2,1)$ to the plane $4x+2y-z=1$ is $\frac{6}{\sqrt{21}}$.