5.4 **Inner product spaces**

In this section, we generalize the scalar product in R^n to other vector spaces.

Definition 1: An **inner product** on a vector space V is an operation on V that assigns to each pair of vectors u and v in V a real number $<u,v>$ satisfying the following conditions:

a. $<u,u>\geq 0$ and $<u,u>>0$ if $u\neq 0$
b. $<u,v>=<v,u>$
c. <αu+βv,z>=α<u,z>+β<u,z> for any $z\in V$

Definition 2: A vector space V with an inner product is called an **inner product space**.

Example 1: Given a vector w with positive entries, for each u and v in R^n, we assign

$$<u,v>=\sum_{i=1}^{n} w_i u_i v_i$$

then it is defined an inner product in R^n. Please check it satisfies the conditions for inner product.

Solution: a. $<u,u>=\sum_{i=1}^{n} w_i u_i u_i=\sum_{i=1}^{n} w_i u_i^2\geq 0$ since $w_i>0$ and $<u,u>=\sum_{i=1}^{n} w_i u_i^2>0$ if $u\neq 0$

b. $<u,v>=\sum_{i=1}^{n} w_i u_i v_i=\sum_{i=1}^{n} w_i v_i u_i=<v,u>$

$$c. <\alpha u+\beta v,z>=\sum_{i=1}^{n} w_i(\alpha u_i+\beta v_i)z_i=\sum_{i=1}^{n} w_i\alpha u_i z_i+\sum_{i=1}^{n} w_i\beta v_i z_i=\alpha\sum_{i=1}^{n} w_i u_i z_i+\beta\sum_{i=1}^{n} w_i v_i z_i=\alpha<u,z>+\beta<v,z>$$

Remark: When $w=<1,1,\cdots,1>$, the above inner product is the scalar product in R^n, we call it the **Standard inner product** for R^n, or $<u,v>=u^T v=\sum_{i=1}^{n} u_i v_i$

Exercise: Given A,B in $R^{m\times n}$, we define the inner product by

(1) $$<A,B>=\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij}$$

Verify that (1) does indeed define an inner product on $R^{m\times n}$.

Example 2: In $C[a,b]$, we define an inner product by the following:

(2) $$<f,g>=\int_a^b f(x)g(x)dx$$

Check (2) satisfies the conditions for inner product.

(a).

$< f, f >= \int_a^b f(x) f(x) dx = \int_a^b f(x)^2 dx \geq 0$ since $\int_a^b f(x)^2 dx =$ area under the curve $y = f(x)^2$ and above x axis from a to b. It's clear that if $f(x) \neq 0$ then $\int_a^b f(x)^2 dx > 0$

(b). $< f, g >= \int_a^b f(x)g(x)dx = \int_a^b g(x) f(x)dx =< g, f >$

(c). $< af + bg, h >= \int_a^b (af(x) + bg(x))h(x)dx = a\int_a^b f(x)h(x)dx + b\int_a^b g(x)h(x)dx = a < f, h > + b < g, h >$

Exercise: Given $w(x) > 0$ for $x \in [a,b]$. Check the following define an inner product in $C[a,b]$,

$$< f, g >= \int_a^b w(x) f(x)g(x)dx$$

The results presented in Section 1 for scalar products in R^n all generalize to inner product spaces. In particular, if v is a vector in an inner product space V, the **length** or **norm** of v is given by

$$\| v \| = \sqrt{< v, v >}$$

Definition 3: Two vectors u and v are called **orthogonal** if $< u, v >= 0$.

Property 1: (The Pythagorean Law) If u and v are orthogonal, then

$$\| u + v \|^2 = \| u \|^2 + \| v \|^2$$

Proof: $\| u + v \|^2 =< u + v, u + v >=< u, u > + < u, v > + < v, u > + < v, v >=< u, u > + < v, v >= \| u \|^2 + \| v \|^2$

Definition 4: If u and v are vectors in an inner product space V and $v \neq 0$, then the **scalar projection of** u **onto** v is given by

$$\alpha = \frac{< u, v >}{\| v \|}$$

And the **vector projection of** u **onto** v is given by

$$proj_v^u = \frac{< u, v >}{< v, v >} v$$

Property 2: If $v \neq 0$, then $u - proj_v^u$ and $proj_v^u$ are orthogonal.

Proof:

$$< u - proj_v^u, proj_v^u > = < u - \frac{<u,v>}{<v,v>}v, \frac{<u,v>}{<v,v>}v > = < u, \frac{<u,v>}{<v,v>}v > - < \frac{<u,v>}{<v,v>}v, \frac{<u,v>}{<v,v>}v > = \frac{<u,v>^2}{<v,v>} - \frac{<u,v>^2}{<v,v>} = 0$$

Property 3 (The Cauchy-Schwarz Inequality). If u and v are vectors in an inner product space V, then

$$|<u,v>| \le \|u\| \ \|v\|$$

Proof. Case 1. If $v=0$, then the above inequality is valid clearly.

Case 2: If $v \neq 0$, then from property 2, we have $u - proj_v^u$ and $proj_v^u$ are orthogonal, by Pythagorean Law, we have $\|u - proj_v^u\|^2 + \|proj_v^u\|^2 = \|u\|^2$

$$\Rightarrow \|proj_v^u\|^2 = \|u\|^2 - \|u - proj_v^u\|^2 \Rightarrow \|proj_v^u\|^2 \le \|u\|^2 \Rightarrow \|\frac{<u,v>}{<v,v>}v\|^2 = \frac{<u,v>^2}{\|v\|^2} \le \|u\|^2 \Rightarrow |<u,v>| \le \|u\| \ \|v\|$$

Definition 4: The unique angle $\theta \in [0,\pi]$ satisfies $\cos\theta = \frac{<u,v>}{\|u\| \ \|v\|}$ is called the angle between two vectors u and v.

Example 3: For the vector space $C[-\pi,\pi]$, we define an inner product

$$< f, g > = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$$

(1). Prove $\cos x$ and $\sin x$ are orthogonal unit vectors.

(2). Using Pythagorean Law find the length of $\sin x + \cos x$

Solution: (1). $< \sin x, \cos x > = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin x \cos x dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin x d \sin x = \frac{1}{2\pi}[\sin^2 x]_{-\pi}^{\pi} = 0$

$$\|\sin x\|^2 = < \sin x, \sin x > = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin x \sin x dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}(1-\cos 2x)dx = \frac{1}{2\pi}[x - \frac{1}{2}\sin 2x]_{-\pi}^{\pi} = 1$$

$$\|\cos x\|^2 = < \cos x, \cos x > = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos x \cos x dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}(1+\cos 2x)dx = \frac{1}{2\pi}[x + \frac{1}{2}\sin 2x]_{-\pi}^{\pi} = 1$$

Thus, $\sin x, \cos x$ are unit vectors, and orthogonal.

(2). Since $\sin x, \cos x$ are orthogonal, by Pythagorean Law, we have

$$\|\sin x + \cos\|^2 = \|\sin x\|^2 + \|\cos x\|^2 = 1 + 1 = 2$$
$$\|\sin x + \cos\| = \sqrt{2}$$

Definition 5: A vector space V is said to be a normed linear space if to each vector $v \in V$ there is associated a real number $\|v\|$, called **norm** of v, satisfying:

(i). $\| v \| \geq 0$, and $\| v \| > 0$ if $v > 0$

(ii). $\| \alpha v \| = | \alpha | \ \| v \|$ for any scalar α

(iii). $\| u + v \| \leq \| u \| + \| v \|$

Theorem 1: If V is an inner product space, then the equation

$$\| v \| = \sqrt{< v, v >}$$

defines a norm on V.

For a given vector space, it is possible to define many different norms. For example, in R^n, we could define

$$\| x \|_1 = \sum_{i=1}^{n} | x_i |$$

$$\| x \|_\infty = \max_{1 \leq i \leq n} | x_i |$$ (called uniform norm or infinity norm)

$$\| x \|_p = \sqrt[p]{\sum_{i=1}^{n} | x_i |^p}$$ (lots of time, we use $\| x \|_2 = \sqrt{\sum_{i=1}^{n} | x_i |^2}$ when $p = 2$)

Exercise: Prove $\| x \|_1 = \sum_{i=1}^{n} | x_i |$ defines a norm in R^n.

Proof: (1).Clearly, for any $x \in R^n$, $\| x \|_1 = \sum_{i=1}^{n} | x_i | \geq 0$ and $\| x \|_1 = \sum_{i=1}^{n} | x_i | > 0$ if $x \neq 0$

(2). $\| \alpha x \|_1 = \sum_{i=1}^{n} | \alpha x_i | = | \alpha | \sum_{i=1}^{n} | x_i | = | \alpha | \ \| x \|_1$

(3). $\| x + y \|_1 = \sum_{i=1}^{n} | x_i + y_i | \leq \sum_{i=1}^{n} (| x_i | + | y_i |) \leq \sum_{i=1}^{n} | x_i | + \sum_{i=1}^{n} | y_i | = \| x \|_1 + \| y \|_1$

Example: Given $x = (1, 4, -1, 2)^T$ in R^4, find $\| x \|_1, \| x \|_2$.

$\| x \|_1 = 1 + 4 + 1 + 2 = 8$

$\| x \|_2 = \sqrt{1 + 16 + 1 + 4} = \sqrt{22}$

HW: 1, 2,3,7,15.