5.6 The Gram-Schmidt Orthogonalization Process

In this section, we wish to form an orthogonal basis for a subspace W .

Example 1: Suppose $W = Span(x_1, x_2)$, where $x_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Find an orthogonal basis $\{v_1, v_2\}$ for W .

Solution: Let $v_1 = x_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

$$proj_{v_1}^{x_2} = \frac{< x_2, v_1 >}{< v_1, v_1 >} v_1 = \frac{-1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$v_2 = x_2 - proj_{v_1}^{x_2} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{-1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ \frac{3}{2} \end{pmatrix}$$

(You can check $< v_1, v_2 >= 0$)

v_1, v_2 are orthogonal basis for W .

In general, we have clear process to transform an ordinary basis $\{x_1, x_2, \cdots, x_n\}$ to orthogonal basis $\{v_1, v_2, \cdots, v_n\}$, and we call this process Gram-Schmidt Orthogonalization Process.

Gram-Schmidt Orthogonalization Process:

Given a basis $\{x_1, x_2, \cdots, x_p\}$ for a subspace W of R^n .

Define

$$v_1 = x_1$$
$$v_2 = x_2 - \frac{< x_2, v_1 >}{< v_1, v_1 >} v_1$$
$$v_3 = x_3 - \frac{< x_3, v_1 >}{< v_1, v_1 >} v_1 - \frac{< x_3, v_2 >}{< v_2, v_2 >} v_2$$
$$\vdots$$
$$v_p = x_p - \frac{< x_p, v_1 >}{< v_1, v_1 >} v_1 - \frac{< x_p, v_2 >}{< v_2, v_2 >} v_2 - \cdots - \frac{< x_p, v_{p-1} >}{< v_{p-1}, v_{p-1} >} v_{p-1}$$

Then $\{v_1, v_2, \cdots, v_p\}$ is an orthogonal basis for W and

$$Span\{x_1, x_2, \cdots, x_p\} = Span\{v_1, v_2, \cdots, v_p\}$$

Example 2: Suppose $W = Span\{x_1, x_2, x_3\}$, and $\left\{ x_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, x_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for W. Find an orthogonal basis for W.

$$v_1 = x_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix},$$

$$v_2 = x_2 - \frac{< x_2, v_1 >}{< v_1, v_1 >} v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} - \frac{5}{14} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{9}{14} \\ \frac{9}{7} \\ -\frac{15}{14} \\ 0 \end{pmatrix}$$

Replace v_2 with $14v_2$: $v_2 = \begin{pmatrix} 9 \\ 18 \\ -15 \\ 0 \end{pmatrix}$

The step to use $14v_2$ replace v_2 is optional step: to make v_2 each component integer, thus it is easier to work with in the next step.

$$v_3 = x_3 - \frac{< x_3, v_1 >}{< v_1, v_1 >} v_1 - \frac{< x_3, v_2 >}{< v_2, v_2 >} v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{14} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} - \frac{9}{630} \begin{pmatrix} 9 \\ 18 \\ -15 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{14} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} - \frac{1}{70} \begin{pmatrix} 9 \\ 18 \\ -15 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ -\frac{2}{5} \\ 0 \\ 1 \end{pmatrix}$$

Rescale(optional): $v_3 = \begin{pmatrix} 4 \\ -2 \\ 0 \\ 5 \end{pmatrix}$

Orthogonal basis for W:

$$\{v_1, v_2, v_3\} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 18 \\ -15 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 0 \\ 5 \end{pmatrix} \right\}.$$

How to find orthonormal basis for subspace W. Orthonormal means the length of each vector is 1.

Example 3: Find a orthonormal basis for W , where W is defined in Example 2.

We have obtained the orthogonal basis for W . We just need to rescale each of following vectors to unit vectors:

$$\{v_1, v_2, v_3\} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 18 \\ -15 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 0 \\ 5 \end{pmatrix} \right\}$$

The orthonormal basis for W is

$$\{u_1, u_2, u_3\} = \left\{ \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{630}} \begin{pmatrix} 9 \\ 18 \\ -15 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{45}} \begin{pmatrix} 4 \\ -2 \\ 0 \\ 5 \end{pmatrix} \right\}$$

Example 4: Consider the vector space $C[-1,1]$ with inner product defined by

$$< f, g >= \int_{-1}^{1} f(x) g(x) dx$$

Find a orthogonal basis for the subspace spanned by 1, x, x^2 .

Solution: Let $x_1 = 1, x_2 = x, x_3 = x^2$.

Let $v_1 = x_1 = 1$.

$$v_2 = x_2 - \frac{< x_2, v_1 >}{< v_1, v_1 >} v_1 = x - \frac{\int_{-1}^{1} x dx}{\int_{-1}^{1} 1 dx} v_1 = x - \frac{0}{2} = x$$

$$v_3 = x_3 - \frac{< x_3, v_1 >}{< v_1, v_1 >} v_1 - \frac{< x_3, v_2 >}{< v_2, v_2 >} v_2 = x^2 - \frac{\int_{-1}^{1} x^2 dx}{\int_{-1}^{1} 1 dx} v_1 - \frac{\int_{-1}^{1} x^3 dx}{\int_{-1}^{1} x^2 dx} v_2 = x^2 - \frac{1}{3} - \frac{3}{4} x$$

$$\text{Orthogonal basis} : 1, x, x^2 - \frac{3}{4} x - \frac{1}{3}$$

HW: 3,4,8.