6.1 Eigenvalues and Eigenvectors

Definition 1: Let A be an $n \times n$ matrix. A scalar λ is called an **eigenvalue** or a **characteristic value** of A if there exists a nonzero vector x such that $Ax = \lambda x$. The vector x is called an **eigenvector** or a **characteristic vector** corresponding to λ.

Example 1: Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}, x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, since $Ax = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda = 3$ is an eigenvalue of A, and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to $\lambda = 3$.

Property: Let λ be an eigenvalue of an $n \times n$ matrix A. The following statements are equivalent.

(a). λ is an eigenvalue of A.

(b). $(A - \lambda I)x = 0$ has a nontrivial solution.

(c). $N(A - \lambda I) \neq \{0\}$

(d). $A - \lambda I$ is singular.

(e). $\det(A - \lambda I) = 0$

Example 2: Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$. Find eigenvalues and eigenvectors of A.

Solution: Let λ be an eigenvalues of A.

$$\det(A - \lambda I) = 0 \Leftrightarrow \begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Leftrightarrow (4-\lambda)(1-\lambda) + 2 = 0$$
$$\Leftrightarrow \lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 3$$

Thus $\lambda_1 = 2, \lambda_2 = 3$ are two eigenvalues of A.

Suppose the eigenvector x of A is corresponding to $\lambda_1 = 2$.

$$(A - \lambda I)x = 0 \Leftrightarrow (A - 2I)x = 0 \Leftrightarrow \begin{bmatrix} 4-2 & -2 \\ 1 & 1-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Leftrightarrow \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Solving this, we have $\begin{cases} x_1 = t \\ x_2 = t \end{cases}$. Let $t = 1$, we have $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda_1 = 2$.

Similarly, we can obtain $x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda_1 = 3$.

Example 2: Given $A=\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. Find eigenvalues and the corresponding eigenvectors.

Solution:

$$|A-\lambda I|=\begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix}=(1-\lambda)^2+4=0 \Rightarrow \lambda_1=1+2i, \lambda_2=1-2i$$

Now, let's find the eigenvector corresponding to $\lambda_1 = 1+2i$,

$$\begin{bmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 1-1-2i & 2 \\ -2 & 1-1-2i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 1 \\ i \end{bmatrix}$$

Similarly, we have $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector corresponding to eigenvalue λ=1-2i

Example 3: Let $\begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$. Find the eigenvalue and the corresponding eigenvector(s) and corresponding eigenspaces.

Solution: $|A-\lambda I|=\begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix}=-\lambda(\lambda-1)^2$

Thus, the eigenvalues are $\lambda_1 = 0, \lambda_2 = \lambda_3 = 1$. The eigenvector(s) corresponding to $\lambda_1 = 0$ can be obtained as before by solving $(A-\lambda I)x=(A-0I)x=Ax=0$.

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solving it, we have $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} t \\ t \\ t \end{bmatrix}=t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The corresponding eigenspace is all vectors of $t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and the corresponding egivenvector is $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}'$.

To find the eigenspace corresponding to $\lambda = 1$, we need to solve the system $(A - I)x = 0$.

$$\left[\begin{array}{ccc|c} 2-1 & -3 & 1 & 0 \\ 1 & -2-1 & 1 & 0 \\ 1 & -3 & 2-1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The eigenspace corresponding to $\lambda = 1$ consists of all vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3t - s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are two independent eigenvectors corresponding to $\lambda = 1$.

HW: 1. (a),(f),(h),3,4,5.