1.3 Convergence Theorems of Fourier Series

In this section, we present the convergence of Fourier series.

An infinite sum is, by definition, a limit of partial sums, that is,

$$\sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) = \lim_{N \to \infty} \sum_{k=1}^{N} a_k \cos(kx) + b_k \sin(kx)$$

provided that the limit exists. Therefore, we say that the Fourier series of $f$ converges to $f$ at the point $x$ if

$$f(x) = \lim_{N \to \infty} S_N(x) = a_0 + \lim_{N \to \infty} \sum_{k=1}^{N} a_k \cos(kx) + b_k \sin(kx)$$

where $S_N(x) = a_0 + \sum_{k=1}^{N} a_k \cos(kx) + b_k \sin(kx)$

With this in mind, we state (without proof) the convergence of Fourier series.

**Theorem 1.** Suppose $f$ is a continuous and $2\pi$-periodic function. Then for each point $x$, where the derivative of $f$ is defined, the Fourier series of $f$ at $x$ converges to $f(x)$, or $|S_N(x) - f(x)| \to 0$ as $N \to \infty$.

Now, we present some variations of the above Theorem. Note that the hypothesis of this theorem requires the function $f$ to be continuous and periodic. However, there are many functions of interest that are neither continuous nor periodic. Before we state the theorem on convergence near a discontinuity, we need the following definition.

**Definition 1:** The left and right limits of $f$ at a point $x$ is defined as follows.

Left limit: $f(x-0) = \lim_{h \to 0^+} f(x-h)$

Right limit: $f(x+0) = \lim_{h \to 0^+} f(x+h)$

The function $f$ is said to be left differentiable at $x$ if the following limit exists:

$$f'(x-0) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

The function $f$ is said to be right differentiable at $x$ if the following limit exists:

$$f'(x+0) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$
Intuitively, $f'(x-0)$ represents the slope of the tangent line to $f$ at $x$ considering only the part of the graph of $y = f(t)$ that lies to the left of $t = x$. The value of $f'(x+0)$ is the slope of the tangent line to $f$ at $x$ considering only the part of the graph of $y = f(t)$ that lies to the right of $t = x$.

Example 1: Let $f(x)$ be the periodic extension of $y = x, -\pi \leq x < \pi$. Then $f(x)$ is discontinuous at $x = \ldots, -\pi, \pi, \ldots$. The left, right limits, left and right derivatives of $f$ at $x = \pi$ are

$$f(\pi - 0) = \pi \quad f(\pi + 0) = -\pi$$
$$f'(\pi - 0) = 1 \quad f'(\pi + 0) = 1$$

Example 2. Let $f(x) = \begin{cases} x, & x \in [0, \pi / 2] \\ \pi - x, & x \in [\pi / 2, \pi] \end{cases}$

The graph of $f$ is the sawtooth wave. This function is continuous, but not differentiable at $x = \pi / 2$. The left and right derivatives at $x = \pi / 2$ are

$$f'(\pi / 2 - 0) = 1 \quad \text{and} \quad f'(\pi / 2 + 0) = -1$$

Now, we are ready to state the convergence theorem for Fourier series at a point where $f$ is not necessarily continuous.

**Theorem 2.** Suppose $f(x)$ is periodic and piecewise continuous. Suppose $x$ is a point where $f$ is left and right differentiable (but not necessarily continuous). Then the Fourier series of $f$ at $x$ converges to
\[ \frac{f(x+0) + f(x-0)}{2} \]

Remark: This theorem stated that at a point of discontinuity of \( f \), the Fourier series of \( f \) converges to the average of the left and right limits of \( f \). At a point of continuity, the left and right limits are the same, and so in this case, Theorem 2 reduces to Theorem 1.

Definition 2: A function is said to be **piecewise smooth** if it is continuous and its derivative is defined everywhere except possibly for a discrete set of points.

For example, the sawtooth function is piecewise smooth since the derivative of \( f \) exists at all points except at multiples of \( \pi/2 \) (which is a discrete set of points).

We say that the Fourier series of \( f(x) \) converges to \( f(x) \) on \([-a,a]\) uniformly if the sequence of partial sums

\[ S_N(x) = a_0 + \sum_{k=1}^{N} a_k \cos(kx) + b_k \sin(kx) \]

converges to \( f(x) \) uniformly as \( N \to \infty \), or \( \max_{-a \leq x \leq a} |f(x) - S_N(x)| \to 0 \) as \( N \to \infty \).

**Theorem 3**: The Fourier series of a piecewise smooth, \( 2\pi \) periodic function \( f(x) \) converges uniformly to \( f(x) \) on \([-\pi,\pi]\).

**Theorem 4**: Suppose \( f \) is an element of \( L^2([-\pi,\pi]) \). Let

\[ S_N(x) = a_0 + \sum_{k=1}^{N} a_k \cos(kx) + b_k \sin(kx) \]

where \( a_n \) and \( b_n \) are the Fourier coefficients of \( f \). Then \( S_N \) converges to \( f \) in \( L^2([-\pi,\pi]) \); that is, \( \|f(x) - S_N(x)\|_{L^2} \to 0 \) as \( N \to \infty \). We also call this **mean** convergence.

Theorem 4 also holds for complex form of Fourier series.

**Theorem 5**: Suppose \( f \) is an element of \( L^2([-\pi,\pi]) \) with (complex) Fourier coefficients given by

\[ \alpha_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int} dt \quad \text{for} \quad n \in \mathbb{Z}. \]

Then the partial sum

\[ S_N(t) = \sum_{k=-N}^{N} \alpha_k e^{int} \]

converges to \( f \) in \( L^2([-\pi,\pi]) \) norm as \( N \to \infty \). In other words, \( \lim_{N \to \infty} \|f - S_N\| \to 0 \).
Energy interpretation. Another way of looking at the theorem is in terms of energy. In signal processing, the integral
\[ \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \|f\|^2 \]
is interpreted as the energy of the "signal" \( f \). The theorem then states that the Fourier series for any "finite energy" function \( f \) converges in the mean to \( f \). The converse is true as well. If the Fourier series converges in the mean to a function, then that function has to have finite energy., i.e., it has to be in \( L^2 ([-\pi, \pi]) \).

Another physical term that is used in connection with Fourier series is frequency mode. In the complex case, this is just one of the terms \( \alpha_k e^{int} \). (The real case is similar.) The energy of a single mode (term) is
\[ \int_{-\pi}^{\pi} |\alpha_k e^{int}|^2 \, dt = 2\pi |\alpha_k|^2 \]

There is a beautiful connection between the energy in a signal and the energy in its modes.

**Parseval’s Theorem.** Suppose \( f \) is an element of \( L^2 ([-\pi, \pi]) \) and its Fourier series is
\[ f(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{int} \]
Then,
\[ \|f\|^2 = \int_{-\pi}^{\pi} |f(t)|^2 \, dt = 2\pi \sum_{k=-\infty}^{\infty} |\alpha_k|^2 \]

Parseval’s Theorem amounts to saying that the energy in a “signal” \( f \) is the sum of the energies in its modes. The real version of the equation in Parseval’s Theorem is
\[ \|f\|^2 = 2\pi |a_0|^2 + \pi \sum_{k=1}^{\infty} \left( |a_k|^2 + |b_k|^2 \right) \]

Remark: Parseval’s theorem can be used in a number of ways. One of them is to obtain sums of series.

Example. Prove \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \).

Solution. From example before, we have the Fourier series for the function \( f(x) = x \) on \([-\pi, \pi]\),
\[ x = \sum_{n=1}^{\infty} 2(-1)^{n+1} \frac{\sin(nx)}{n} , \quad x \in [-\pi, \pi] \]. (Note that \( b_n = \frac{2(-1)^{n+1}}{n} \))

By Parseval’s theorem,
\[ \|x\|^2 = \pi \sum_{n=1}^{\infty} \left( \frac{2(-1)^{n+1}}{n} \right)^2 = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2} \]
where \( \|x\|^2 = \int_{-\pi}^{\pi} x^2 \, dx = \frac{2}{3} \pi^3 \).
Thus, we have

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \]

Question: Prove \[ \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96} \].

Hint: Please consider: The Fourier series for \( f \) on the interval \( 0 \leq x \leq \pi \) given by

\[ f(t) = \begin{cases} 
  x & \text{if } 0 \leq x \leq \pi/2 \\
  \pi - x & \text{if } \pi/2 \leq x \leq \pi 
\end{cases} \]

and extends to the interval \( -\pi \leq x \leq 0 \) as an even function.