Fourier Transform of a Convolution

Definition: Suppose f and g are two square-integrable functions. The **convolution** of f and g, denoted $f * g$, is defined by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-x)g(x)dx$$

The preceding definition is equivalent to

$$(f * g)(t) = \int_{-\infty}^{\infty} f(x)g(t-x)dx$$

We have the following theorem on the Fourier transform of a convolution of two functions.

Theorem: Suppose f and g are two square-integrable functions. Then

$$F[f * g] = \sqrt{2\pi} F[f] \cdot F[g]$$
$$F^{-1}\{F[f] \cdot F[g]\} = \frac{1}{\sqrt{2\pi}} f * g$$

Proof. To derive the first equation, we use the definitions of the Fourier transform and convolution:

$$\begin{aligned}
F[f * g](\lambda) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f * g)(t) e^{-i\lambda t} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t-x)g(x)dx e^{-i\lambda t} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t-x)e^{-i\lambda(t-x)} g(x)e^{-i\lambda x} dxdt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t-x)e^{-i\lambda(t-x)} dt g(x)e^{-i\lambda x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s)e^{-i\lambda s} ds \int_{-\infty}^{\infty} g(x)e^{-i\lambda x} dx \qquad (\text{Let } s = t - x) \\
&= \sqrt{2\pi} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s)e^{-i\lambda s} ds \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)e^{-i\lambda x} dx \right) \\
&= \sqrt{2\pi} F(f) F(g)
\end{aligned}$$

$$F[f] \cdot F[g] = \frac{1}{\sqrt{2\pi}} F[f * g]$$

$$F^{-1}\{F[f] \cdot F[g]\} = F^{-1}\left\{ \frac{1}{\sqrt{2\pi}} F[f * g] \right\} = \frac{1}{\sqrt{2\pi}} F^{-1}\{F[f * g]\} = \frac{1}{\sqrt{2\pi}} f * g$$

Adjoint of the Fourier Transform

Definition: The adjoint of a linear operator $T:V \to W$ between two inner product spaces is an operator $T^*:W \to V$ such that

$$< v, T^*(w) >_V = < T(v), w >_W$$

In the following theorem, we show that the adjoint of the Fourier transform is the inverse of the Fourier transform.

Theorem: Suppose f and g are two square-integrable functions. Then

$$< F[f], g >_{L^2} = < f, F^{-1}[g] >_{L^2}$$

Proof. We have

$$\begin{aligned}
< F[f], g >_{L^2} &= \int_{-\infty}^{\infty} F(f)(\lambda)\overline{g(\lambda)}d\lambda \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt\,\overline{g(\lambda)}d\lambda \\
&= \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{g(\lambda)e^{i\lambda t}}d\lambda dt \quad \left(\text{notice: } \overline{F^{-1}(g)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{g(\lambda)e^{i\lambda t}}d\lambda\right) \\
&= < f, F^{-1}[g] >_{L^2}
\end{aligned}$$

Plancherel Formula

The Plancherel Formula states that the Fourier transform preserves the L^2 inner product.

Theorem: Suppose f and g are two square-integrable functions. Then

$$< F[f], F[g] >_{L^2} = < f, g >_{L^2}$$
$$< F^{-1}[f], F^{-1}[g] >_{L^2} = < f, g >_{L^2}$$

In particular, $\|F[f]\|_{L^2} = \|f\|_{L^2}$

Proof.

$$< F[f], F[g] >_{L^2} = < f, F^{-1}F[g] >_{L^2} = < f, g >_{L^2}$$
$$< F^{-1}[f], F^{-1}[g] >_{L^2} = < F\left\{F^{-1}[f]\right\}, F\left\{F^{-1}[g]\right\} >_{L^2} = < f, g >_{L^2}$$
$$\|F[f]\|_{L^2}{}^2 < F[f], F[f] >_{L^2} = < f, f >_{L^2} = \|f\|_{L^2}{}^2 \Rightarrow \|F[f]\|_{L^2} = \|f\|_{L^2}$$

Remark: The equation $\|F[f]\|_{L^2} = \|f\|_{L^2}$ is analogous to Parseval's equation.

Example: Evaluate $\int_{-\infty}^{\infty} \frac{\sin^2(\lambda\pi)}{(9-\lambda^2)^2} d\lambda$

Solution:

From the example in the previous notes, we have that for $f(\mathrm{t}) = \begin{cases} \sin 3t & \text{if } -\pi \le t \le \pi \\ 0 & \text{otherwise} \end{cases}$, its Fourier transform is

$$F[f](\lambda) = \frac{-i3\sqrt{2}\sin(\lambda\pi)}{\sqrt{\pi}(9-\lambda^2)}$$

Thus, by Plancherel Formula, $\|F[f]\|_{L^2} = \|f\|_{L^2}$, we have

$$\|F[f]\|^2 = < F[f](\lambda), F[f](\lambda) > = \int_{-\infty}^{\infty} F[f](\lambda) \cdot \overline{F[f](\lambda)} d\lambda = \frac{18}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2(\lambda\pi)}{(9-\lambda^2)^2} d\lambda$$

$$\|f\|^2 = < f, f > = \int_{-\infty}^{\infty} f(t) \cdot \overline{f(t)} dt = \int_{-\pi}^{\pi} \sin^2 3t dt = \pi$$

$$\frac{18}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2(\lambda\pi)}{(9-\lambda^2)^2} d\lambda = \pi$$

$$\int_{-\infty}^{\infty} \frac{\sin^2(\lambda\pi)}{(9-\lambda^2)^2} d\lambda = \frac{\pi^2}{18}$$

Question 1: Evaluate $\int_{-\infty}^{\infty} \frac{\sin^2(\lambda)}{\lambda^2} d\lambda$ (answer: π^2)

hint: consider the following function and its Fourier Transform, and use Plancherel Formula.

$$f(\mathrm{t}) = \begin{cases} 1 & \text{if } -\pi \le t \le \pi \\ 0 & \text{otherwise} \end{cases} \qquad F[f](\lambda) = \sqrt{\frac{2}{\pi}} \frac{\sin(\pi\lambda)}{\lambda}$$

Question 2: Prove $\int_{-\infty}^{\infty} \frac{\sin(\pi\lambda)}{\lambda(1+\lambda^2)} d\lambda = \pi(1-e^{-\pi})$

$$f(\mathrm{t}) = \begin{cases} 1 & \text{if } -\pi \le t \le \pi \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g(\mathrm{t}) = \begin{cases} e^{-t} & \text{if } 0 \le t \\ 0 & \text{otherwise} \end{cases}$$

Hint: $F[f](\lambda) = \sqrt{\frac{2}{\pi}} \frac{\sin(\pi\lambda)}{\lambda}$ and $F[g](\lambda) = \frac{1}{\sqrt{2\pi}(1+\lambda i)}$

using $< f, g > = < F[f], F[g] >$