4.1 Haar Wavelets

Wavelets were first applied in geophysics to analyze data from seismic surveys, which are used in oil and mineral exploration to get “pictures” of layering in subsurface rock. In fact, geophysicists rediscovered them; mathematicians had developed them to solve abstract problems some twenty years earlier but had not anticipated their applications in signal processing.

The key to an accurate seismic survey is a proper analysis of each trace. The Fourier transform is not a good tool here. It can only provide frequency information (the oscillations that comprise the signal). It does not provide direct information about when an oscillation occurred. Wavelets can keep track of time and frequency information.

![Standard seismic trace](image)

**Figure 1** A typical seismic trace. Displacement is plotted versus time. Both the oscillations and the time they occur are important.

**Haar Wavelet**

**The Haar Scaling Function**

There are two functions that play a primary role in wavelet analysis, the scaling function \( \phi \) and the wavelet \( \psi \). These two functions generate a family of functions that can be used to break up or reconstruct a signal. To emphasize the “marriage” involved in building this “family”, \( \phi \) is sometimes called the “father wavelet” and \( \psi \), the “mother wavelet”.

The simplest wavelet analysis is based on the Haar scaling function, whose graph is given in Figure 2.
Definition 1. The Haar scaling function is defined as

\[ \phi(x) = \begin{cases} 
1, & \text{if } 0 \leq x < 1 \\
0, & \text{elsewhere}
\end{cases} \]

The building blocks are translations and dilations (both in height and width) of this basic graph.

We wish to illustrate the basic ideas involved in such an analysis. Consider the signal shown below.

Fig. 3 Voltage from a faulty meter
We may think of this as a measurement of some physical quantity—perhaps line voltage over a single cycle—as a function of time. The two sharp spikes in the graph might represent noise coming from a loose connection in the voltmeter, and we want to filter out this undesirable noise. The graph in Figure 4 shows one possible approximation to the signal using Haar building blocks. The high frequency noise shows up as tall, thin block. An algorithm that deletes the thin blocks will eliminate the noise and not disturb the rest of the signal.

The building blocks generated by the Haar scaling function are particularly simple and illustrate the general ideas underlying a multiresolution analysis, which we will discuss in detail. The disadvantage of the Haar wavelets is that they are discontinuous and therefore do not approximate continuous signals very well.

The function $\phi(x - k)$ has the same graph as $\phi$ but translated to the right by $k$ units (assuming $k$ is positive). Let $V_0$ be the space of all functions of the form

$$\sum_{k \in \mathbb{Z}} a_k \phi(x - k), \quad a_k \in \mathbb{R}$$

where $k$ can range over any finite set of positive or negative integers. Since $\phi(x - k)$ is discontinuous at $x = k$ and $x = k + 1$, an alternative description of $V_0$ is that it consists of all piecewise constant functions whose discontinuities are contained in the set of integers. Since $k$ ranges over a finite set, each element of $V_0$ is zero outside a bounded set. Such a function is said to have finite or compact support. The graph of a typical element of $V_0$ is given in the following graph.
Graph of typical element in $V_0$

Note that a function in $V_0$ may not have discontinuities at all the integers (for example, if $a_1 = a_2$, then the preceding sum is continuous at $x = 2$).

Example 1: Graph $f(x) = 2\phi(x) + 3\phi(x-1) + 3\phi(x-2) - \phi(x-3)$

Graph of $\phi(2x)$
Example 2: Graph of function \( f(x) = 4\phi(2x) + 2\phi(2x-1) + 2\phi(2x-2) - \phi(2x-3) \)

**Definition 2.** Suppose \( j \) is any nonnegative integer. The space of step functions at level \( j \), denoted by \( V_j \), is defined to be the space spanned by the set
\[
\{ \cdots, \phi(2^j x + 1), \phi(2^j x), \phi(2^j x - 1), \phi(2^j x - 2), \cdots \}
\]
over the real numbers. \( V_j \) is the space of piecewise constant functions of finite support whose discontinuities are contained in the set
\[
\{ \cdots, -\frac{2}{2^j}, -\frac{1}{2^j}, 0, \frac{1}{2^j}, \frac{2}{2^j}, \frac{3}{2^j}, \cdots \}
\]
A function in \( V_0 \) is a piecewise constant function with discontinuities contained in the set of integers. Any function in \( V_0 \) is also contained in \( V_j \), which consists of piecewise constant functions whose discontinuities are contained in the set of half-integers \( \{ \cdots, 1/2, 0, 1/2, 1, 3/2, \cdots \} \). The same applies for \( V_1 \subset V_2 \) and so forth:
\[
V_0 \subset V_1 \subset \cdots \subset V_{j-1} \subset V_j \subset V_{j+1} \subset \cdots
\]
This containment is strict. For example, the function \( \phi(2x) \) does not belong to \( V_0 \).

**Question:** Prove \( V_0 \subset V_1 \subset \cdots \subset V_{j-1} \subset V_j \subset V_{j+1} \subset \cdots \)

**Basic Properties of the Haar Scaling Function.**

**Theorem 1.**

(a) A function \( f(x) \) belongs to \( V_0 \) \( \iff \) \( f(2^j x) \) belongs to \( V_j \)

(b) A function \( f(x) \) belongs to \( V_j \) \( \iff \) \( f(2^{-j} x) \) belongs to \( V_0 \)
Proof. (a). if \( f(x) \) belongs to \( V_0 \), then \( f(x) = \sum_{k \in \mathbb{Z}} a_k \phi(x - k) \) \( a_k \in \mathbb{R} \), so
\[
f(2^j x) = \sum_{k \in \mathbb{Z}} a_k \phi(2^j x - k) \quad a_k \in \mathbb{R},
\]
this means that \( f(2^j x) \) belongs to \( V_j \).

For (b), we can similarly prove.

**Theorem 2.** The set of functions \( \{ 2^{j/2} \phi(2^j x - k), k \in \mathbb{Z} \} \) is an orthonormal basis of \( V_j \).

Question: Prove Theorem 2.

**The Haar Wavelet**

Having an orthonormal basis of \( V_j \) is only half of the picture. In order to solve our noise-filtering problem, we need to have a way of isolating the “spikes” that belong to \( V_j \) but that are not members of \( V_{j-1} \). This is where the wavelet \( \psi \) enters the picture.

The idea is to decompose \( V_j \) as an orthogonal sum of \( V_{j-1} \) and its complement. Again, let start with \( j = 1 \) and identify the orthogonal complement of \( V_0 \) in \( V_1 \). Since \( V_0 \) is generated by \( \phi \) and its translates, it is reasonable to expect that the orthogonal complement of \( V_0 \) is generated by the translates of some function \( \psi \). We need \( \psi \in V_1 \), and \( \psi \) is orthogonal to \( V_0 \). Consider the following function.

**Definition 3.** The **Haar wavelet** is the function
\[
\psi(x) = \phi(2x) - \phi(2x - 1)
\]

The graph of Haar wavelet is below.

Question: Prove \( \psi \in V_1 \), and \( \psi \) is orthogonal to \( V_0 \).
We find out that any function
\[ f_1(x) = \sum_{k \in \mathbb{Z}} a_k \phi(2x-k) \in V_1 \]
is orthogonal to \( V_0 \), that is, orthogonal to each \( \phi(x-l), l \in \mathbb{Z} \) if and only if
\[ a_1 = -a_0, a_3 = -a_2, \cdots \]
(prove the above result as an exercise)

Solution:
\[
f_1(x) = \sum_{k \in \mathbb{Z}} a_k \phi(2x-k) \perp V_0 \rightarrow \sum_{k \in \mathbb{Z}} a_k \phi(2x-k) \perp \phi(x) \quad \text{(since } \phi(x) \in V_0 \text{)} \rightarrow \left( \sum_{k \in \mathbb{Z}} a_k \phi(2x-k), \phi(x) \right) = 0
\]
\[ \phi(x) = \phi(2x) + \phi(2x-1) \]
\[ \left( \sum_{k \in \mathbb{Z}} a_k \phi(2x-k), \phi(2x) + \phi(2x-1) \right) = 0 \]
\[ \left( \sum_{k \in \mathbb{Z}} a_k \phi(2x-k), \phi(2x) + \phi(2x-1) \right) = a_0 \left( \phi(2x), \phi(2x) \right) + a_1 \left( \phi(2x-1), \phi(2x-1) \right) = a_0 / 2 + a_1 / 2 = 0 \rightarrow a_0 = -a_1 \]

Similarly, \[ \left( \sum_{k \in \mathbb{Z}} a_k \phi(2x-k), \phi(x-1) \right) = 0 \]
\[ \left( \sum_{k \in \mathbb{Z}} a_k \phi(2x-k), \phi(2x-2) + \phi(2x-3) \right) = a_2 \left( \phi(2x-2), \phi(2x-2) \right) + a_3 \left( \phi(2x-3), \phi(2x-3) \right) = a_2 / 2 + a_3 / 2 = 0 \rightarrow a_2 = -a_3 \]

.......

In this case, we have
\[ f_1(x) = \sum_{k \in \mathbb{Z}} a_{2k} \phi(2x-k) - \phi(2x-k-1) = \sum_{k \in \mathbb{Z}} a_{2k} \psi(x-k) \]

In other words, a function in \( V_1 \) is orthogonal to \( V_0 \) if and only if it is of the form \( \sum_{k \in \mathbb{Z}} a_k \psi(x-k) \)
(relabeling \( a_{2k} \) by \( a_k \)).

**Definition 4.** Let \( U \) and \( V \) be subspaces of \( W \). Then \( W \) is said to be the direct sum of \( U \) and \( V \), and we write \( W = U \oplus V \) if \( W = U + V \) and \( U \cap V = \{0\} \).

Let \( W_0 \) be the space of all functions of the form
\[ \sum_{k \in \mathbb{Z}} a_k \psi(x-k) \quad a_k \in \mathbb{R} \]

where, again, we assume that only a finite number of the \( a_k \) are nonzero. What we have just shown is that \( W_0 \) is the orthogonal complement of \( V_0 \) in \( V_1 \) or, in other words, \( V_1 = V_0 \oplus W_0 \).
In a similar manner, the following more general result can be established.

**Theorem 3.** Let \( W_j \) be the space of functions of the form

\[
\sum_{k \in \mathbb{Z}} a_k \psi(2^j x - k) \quad a_k \in \mathbb{R}
\]

assume that only a finite number of the \( a_k \) are nonzero. \( W_j \) is the orthogonal complement of \( V_j \) in \( V_{j+1} \) and

(2) \[ V_{j+1} = V_j \oplus W_j \]

**Proof.** To show (2), we have to prove two facts:

1. Each function in \( W_j \) is orthogonal to every function in \( V_j \).
2. Any function in \( V_{j+1} \) that is orthogonal to \( V_j \) must belong to \( W_j \).

For the first requirement, suppose that \( g(x) = \sum_{k \in \mathbb{Z}} a_k \psi(2^j x - k) \in W_j \) and \( f \in V_j \), we need to show

\[
< g, f >_{L^2} = \int_{-\infty}^{\infty} g(y) \overline{f(y)} dy = 0
\]

Since \( f \in V_j \), the function \( f(2^{-j} x) \in V_0 \). So

\[
< g, f >_{L^2} = \int_{-\infty}^{\infty} g(y) \overline{f(y)} dy = \int_{-\infty}^{\infty} \sum_{k \in \mathbb{Z}} a_k \psi(2^j y - k) \overline{f(y)} dy
\]

\[
= 2^j \int_{-\infty}^{\infty} \sum_{k \in \mathbb{Z}} a_k \psi(y - k) \overline{f(2^{-j} x)} dx
\]

\[
= 0 \quad \text{(because } \psi \text{ is orthogonal to } V_0 \text{)}
\]

Thus, \( g \) is orthogonal to any \( f \in V_j \) and thus we have proved the first one requirement.

The discussion leading to equation (1) established the second requirement when \( j = 0 \) since we proved that any function in \( V_1 \) that is orthogonal to \( V_0 \) must be a linear combination of \( \{ \psi(x - k), k \in \mathbb{Z} \} \). The argument of general \( j \) is very similar to the case when \( j = 0 \).

By successively decomposing \( V_j, V_{j-1} \) and so on, we have

\[
V_j = W_{j-1} \oplus V_{j-1}
\]

\[
= W_{j-1} \oplus W_{j-2} \oplus V_{j-2}
\]

\[
= W_{j-1} \oplus W_{j-2} \oplus W_0 \oplus V_0
\]
So each \( f \in V_j \) can be decomposed uniquely as a sum

\[
f = w_{j-1} + w_{j-2} + \cdots + w_0 + v_0
\]

where \( w_k \in W_k, 0 \leq k \leq j-1, \) and \( v_0 \in V_0 \)

Intuitively, \( w_j \) represents the “spikes” of \( f \) of width \( 1/2^{j+1} \) that can’t be represented as linear combinations of spikes of other widths.

When \( j \to \infty \), we have the following:

**Theorem 4.** The space \( L^2(R) \) can be decomposed as an infinite orthogonal direct sum

\[
L^2(R) = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots
\]

In other words, each \( f \in L^2(R) \) can be written uniquely as

\[
f = v_0 + \sum_{j=0}^{\infty} w_j
\]

where \( v_0 \in V_0 \) and \( w_k \in W_k \).

The infinite sum should be thought of as a limit of finite sums. In other words,

\[
f = v_0 + \lim_{N \to \infty} \sum_{j=0}^{N} w_j
\]

where the limit is taken in the sense of \( L^2 \).

The proof is beyond the scope of this course.